Bankruptcy Prediction Based on First-Passage Models with Markovian Credit Migration

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In this paper, we develop a bankruptcy prediction framework that could be used in the study of rating systems. A hybrid model with credit migration is proposed, which could not only capture the differences in bankruptcy probabilities of firms with different ratings, but also of firms within the same rating. Moreover, we derive explicitly computable approximations for the expected bankruptcy time and the tail probabilities of the bankruptcy time, and further applied them to the evaluation of rating systems' performances. Numerical results are also presented.

1 Introduction

In risk management, credit rating systems provide valuable information by dividing firms into different ratings, and indicate that firms in distinct ratings should have different bankruptcy probability. However, firms within the same rating are still different in bankruptcy probabilities, which are not captured by the rating information. Moreover, if this difference is relatively large, the rating system would only provide little information to a firm's bankruptcy probability, and thus be less efficient. This paper studies this issue by a three steps progress. First, we formulate a bankruptcy prediction framework that can not only reflect the differences in bankruptcy probabilities of firms with distinct credit ratings, but also of firms with the same rating but with distinct individual characteristics. Second, we derive explicitly computable approximations of the expected bankruptcy time and the tail probabilities of the bankruptcy time. And last, we use these approximations to examine the proportion of bankruptcy probabilities that is determined by the credit rating system, which could be used to evaluate the rating system's performance.

In the first step, we formulate our credit risk model. To capture both the difference in bankruptcy probabilities among firms of distinct ratings and of the same rating, we need to have a model which the bankruptcy probability is constructed by two components: an exogenous part that is the same among all firms with the same rating, and an endogenous part that may vary among firms even with the same rating [1]. For the exogenous part, we incorporate the Markov chain model proposed by Jarrow et al.(1997,)

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[1] For the terminologies of exogenous and endogenous bankruptcies, see Acharya and Carpenter (2002) for example.

which allows us to have different bankruptcy probabilities for firms with distinct ratings via the transition probabilities matrix. For the endogenous part, on the other hand, we use the standard barrier-option framework, which enable us to have different bankruptcy probabilities among firms via different firm value.

To be more precise, our framework is based on two useful phenomena. First, we apply credit rating as the underlying Markov chain and using bankruptcy state to formulate exogenous bankruptcy. This setting can be dated back to Jarrow et al. (1997), in which they formulate the credit rating process as a K-state Markov chain, with state 1 to $K-1$ represents different credit ratings, and state K be the default state. The securities are then valued as standard options combined with an exogenous default time depending only on the time when the Markov chain goes into the default state. Further researches along this line can be found as follows: Kijima and Komoribayashi (1998) relaxes the assumption on the transformation of the transition probability matrix from the real probability measure to the risk-neutral measure; Das and Tufano (1996) extends the model to the case of random recovery rates; Lando (1998) elaborates it to a conditionally Markov model; and Bielecki and Rutkowski (2000) takes the stochastic interest rate into consideration. Note that all of these models successfully describe the linkage between credit rating and the exogenous bankruptcy, but fail to explain why different firms with the same credit rating would have different bankruptcy probability since they did not consider endogenous bankruptcy.

Second, we release the independency between the firm value and the rating process, and incorporate the bankruptcy barrier approach to formulate the endogenous bankruptcy. That is, we use bankruptcy barrier to formulate endogenous bankruptcy. Without using the credit rating process, this setting can be found in Brockman and Turtle (2003), in which they treat corporate securities as barrier options acting on the firm-value process, and regard the bankruptcy as the first-passage time $\tau^{en}(H)$ relative to an exogenous constant barrier H. They further explore this idea by providing empirical evidences that the implied barriers are significant in the securities market. Here, we adopt this framework to model the endogenous bankruptcy, with further incorporation of the credit rating information as mentioned above. Other considerations could also be brought in, such as the optimal capital structure and finite maturity debts [e.g. Leland et al. (1996)] and stochastic interest rate [e.g. Longstaff and Schwartz (1995)].

In summary, we use a Markov random walk[2] framework, in which the firm's observable rating process is incorporated as the underlying Markov chain, and the logarithm of the firm value is modeled as a random walk, such that the increment distribution is determined by the current rating. The exogenous and endogenous bankruptcies are then modeled through the bankruptcy state approach and the knock-out barrier approach, respectively and simultaneously. In other words, the bankruptcy time is decomposed as two parts. An exogenous bankruptcy is determined by the firm's current credit rating, to which the bankruptcy probability is shared across all firms with the same rating; and an endogenous bankruptcy is determined by both the rating and the firm-value process, to which the bankruptcy probability is govern by the individual firm alone. As a result, our model is capable of capturing the following features: the differences in bankruptcy

[2]A Markov random walk is a random walk with the increment distribution determined by the state of the underlying Markov chain.

probabilities for firms in different ratings by making the bankruptcy rating dependent; the differences for firms in the same rating by allowing the endogenous bankruptcy to vary across firms with distinct firm values; and the effects of rating transition on the bankruptcy probabilities by having different transition probabilities and increment distributions under different rating. Therefore, by having different current firm values, corporations with same ratings can have different bankruptcy probabilities under our model.

Moreover, from modeling point of view, we propose a hybrid-type model that not only enhances the structural-form model by bring in the credit rating information, but also strengthen the reduced-form model by taking care of both endogenous and exogenous bankruptcies. This is also a further improvement in the existing hybrid approaches such as Madan and Unal (1998), who formulates the default intensity as a function of the firm value. Here we completely removing the concept of intensity and directly make the linkage between bankruptcies and firm values. Nevertheless, as in Jarrow et al. (1997), this framework can be modified into continuous- or discrete-time model. However, note that the firm value is often unobservable, researchers commonly use the quantity of book value of assets less the book value of share holders' equity, plus the market value of equity, as a proxy of firm value; see Brockman and Turtle (2003) for example. This quantity, as well as the rating transition information, is in discrete time. Furthermore, discretization is also required for numerical computations and simulations. Hence, in this paper, we will study discrete time model only.

Next, in the second part, we provide explicitly computable approximations to the expected value and tail probability of the endogenous bankruptcy time under this framework. The former can be done by estimating the expected value of τ^{en}, the first-passage time of the firm value process crossing a constant barrier $H < 0$. We will provide two approximations of doing so. The credit-independent approximation is simply done by estimating the expected first-passage time $E_i(\tau^{en})$ under a given initial state (credit rating) i by the expected first-passage time $E_\pi(\tau^{en})$ under the invariant measure π. And we approximate $E_\pi(\tau^{en})$ by:

$$E_\pi(\tau^{en}) \approx \frac{H}{\mu_\pi}, \tag{1}$$

where $\mu_\pi < 0$ is the mean of the Markov random walk under the invariant measure π. The credit-independent approximation is relatively easy to compute, and can be more accurate when $|H|$ is sufficiently large.

However, the credit-independent approximation has two shortcomings. First, it does not take the initial state into the account; different initial rating will result in identical approximations for the expected bankruptcy time. This makes it unsuitable for the study of credit risk and credit rating systems since one might want to see the differences in implied bankruptcy probabilities due to distinct initial ratings. Second, the credit-independent approximation is less accurate when the barrier H is not far, which can be vital for credit-related securities valuations. For example, for credit derivatives such as CDO and CDS, whether the bankruptcy occurs before the mature date or not, will dramatically change the cash flow, and hence the value of the securities. Even a one-day difference near the maturity would cause critical changes. Therefore, it is crucial to have more accurate estimate for bankruptcy prediction.

To this end, we provide a credit-dependent approximation of the expected first-passage time $E_i(\tau^{en})$, which can reflect different expected bankruptcy time under distinct initial rating and give more accurate estimate. This approximation is based on Wald's equation for Markov random walks given by Fuh and Lai (1998), in which the estimate involves an expected overshoot term ρ_- and an initial distribution related term; see Proposition 1 in Section 3 for details. This overshoot term is not easy to compute numerically in Markov random walks, however it can be explicitly computed through Spitlzer's formula in simple random walks. The mathematics contribution in this paper is that we provide an approximation of the overshoot ρ_- under Markov random walks by the corresponding terms under simple random walks. The accuracy is assured when the transition probability matrix is near diagonal, in which it is also the property in credit rating process. By using the same technique, we have approximations for the tail probabilities of the bankruptcy time, in which the equality between the tail probabilities and the overshoot term ρ_- is provided by Fuh (2004) instead.

Last, in the third part, we use these approximations to examine the proportion of bankruptcy probability that is determined by the rating, which provides a method to evaluate the performance of a rating system. Notice that the rating agencies would like to keep the number of rating levels at a minimum to be efficient and simplified, but at the same time facing the problem that when there are too less rating levels, the difference within a rating would be too large such that one could know only little information about the firm's bankruptcy probability when given the firm's rating, making the rating information irrelevant. This would force the agency to increase the number of rating levels, which has been undergone by many rating agencies. Therefore, an optimal number of rating levels would be an interest of these agencies. Thus, we provide a method to evaluate the performance of a rating system when having given number of rating level through the approximations we derived, which is done by examine the proportion of bankruptcy probability that is determined by the rating information alone. The larger the proportion is, the stronger the rating information could determine the bankruptcy probability, and a better performance the rating system functions. We demonstrate this application with numerical illustrations.

The rest of the paper is organized as follows. Section 2 gives the model formulation. Section 3 provides analytic approximations of the expected value and tail probabilities of the bankruptcy time under this framework, and propose the application in evaluating rating system performance. Three numerical results are given in Section 4 to illustrate our methods. Section 5 concludes. The proofs are deferred to Appendix.

2 The Model

The main structure of our framework is to formulate the firm value process by a Markov random walk, where the underlying Markov chain being the firm's credit rating process. The exogenous bankruptcy is then be triggered when the Markov chain goes into a bankruptcy state, while the endogenous bankruptcy is defined as the first passage time of the firm value process crosses a constant exogenous bankruptcy barrier. For consistency, the Markovian credit rating process will be followed in the framework of Jarrow et al.

(1997).

2.1 Model Formulation

To begin with, we consider a discrete-time, friction-less economy on a finite time horizon $[0, t_{max}]$, with a default-free money market account B_t and the risk corporate zero-coupon bonds being traded, and with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le t_{max}}, P)$. We assume that the markets for these corporate bonds are complete and arbitrage-free; this is equivalent to assuming an unique equivalent martingale measure that makes the price processes of these securities martingales after discounting by the money market account. See Harrison and Pliska (1981) for details.

Now, we begin our model formulation. First, we let $V(t,T)$ be the price of the corporate zero-coupon bond that pays one dollar if the bankruptcy has not occur at maturity time T, and pays δ instead if the firm is bankruptcy before T, with $t < T$ and $0 \le \delta < 1$. Let τ be the random time at which the bankruptcy occurs. Then, under the equivalent martingale measure,

$$V(t,T) = E\left(\frac{B_t}{B_T}(\delta 1_{\{\tau \le T\}} + 1_{\{\tau > T\}})|\mathcal{F}_t\right). \tag{2}$$

Second, we assume that the firm value process follows a Markov random walk based on the firm's credit-rating process. That is, let J_t be the credit rating of the firm at time t, which is formed as a discrete time homogeneous Markov chain on a finite state space $\mathcal{S} = \{1, \ldots, K\}$, where states 1 to $K-1$ represent the possible credit ratings, and the state K represents the bankruptcy state. The transition probability matrix of this Markov chain is given by

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1K} \\ q_{21} & q_{22} & \cdots & q_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ q_{(K-1)1} & q_{(K-1)2} & \cdots & q_{(K-1)K} \\ 0 & 0 & \cdots & 1 \end{pmatrix} \tag{3}$$

under the equivalent martingale measure, with q_{ij} represents the probability for an i-rating firm at time t to a j-rating firm at time $t+1$. Note that in the Markov chain with transition probability matrix (3), the bankruptcy state K is an absorbing state. Next, for given the Markov chain J_t, assume that the logarithm of the firm value process V_t follows a random walk

$$\ln \frac{V_t}{V_0} := S_t = \sum_{n=1}^{t} X_n, \tag{4}$$

where the distribution of X_n depends only on J_n. That is, $X_n|_{J_n=j}$ follows a certain distribution F_j and is independent of other factors.

Last, assume that the bankruptcy time τ is triggered by two types of bankruptcies. The exogenous bankruptcy time, τ^{ex}, is defined as the first time when the credit-rating process reaches the bankruptcy state K,

$$\tau^{ex} := \tau^{ex}(K) = \min\{n : J_n = K, n > 0\}. \tag{5}$$

And the endogenous bankruptcy time, τ^{en}, is defined as the first-passage time of the firm value process crosses an exogenous constant bankruptcy barrier H,

$$\tau^{en} := \tau^{en}(H) = \min\{n : S_n < H, n > 0\}. \tag{6}$$

The bankruptcy time τ is then defined to be the smaller one of these two

$$\tau = \min\{\tau^{ex}, \tau^{en}\}. \tag{7}$$

Notice that our setting is a generalization of Jarrow et al. (1997) and Brockman and Turtle (2003) by considering both exogenous and endogenous bankruptcies at the same time; namely, we construct a hybrid version that elaborate these two models. When the bankruptcy barrier tends to negative infinity, the model degenerates to Jarrow et al.'s case as the endogenous bankruptcy is removed. When the state space degenerates to one state only, our model becomes Brockman and Turtle's case as the credit-rating process is excluded.

As that in Jarrow et al. (1997), this model can be easily modified to the continuous time case. Simply change the state process J_t to a continuous-time Markov chain with transition probability matrix given by

$$Q_t = exp(t\Lambda) = \sum_{k=0}^{\infty}(t\Lambda)^k/k!, \tag{8}$$

where Λ is the $K \times K$ generator matrix

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \cdots & \Lambda_{1K} \\ \Lambda_{21} & \Lambda_{22} & \cdots & \Lambda_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ \Lambda_{(K-1)1} & \Lambda_{(K-1)2} & \cdots & \Lambda_{(K-1)K} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{9}$$

and replace $\ln V_t = S_t = \int_0^t dX_z$, where the distribution of dX_z depends only on J_z. That is, $dX_z|_{J_z=j}$ follows F_j distribution and is independent of other factors.

2.2 Discussions

Several comments are given in this subsection. First, empirical studies show that the transition probability matrix of the credit-rating is nearly diagonal [cf. Jarrow et al. (1997)]; this assures the accuracy of our estimations provided in Section 3 below. Second, the assumption of "time-homogeneous" is purely for simplicity. Studies after Brockman and Turtle (2003) suggest non-time-homogeneous or even non-Markov model for credit rating process; see Frydman and Schuermann (2008) for example. Our framework can be easily modified to these changes and as one can see later on. Third, the order of the states is irrelevant in this framework, namely, letting state 1 be the best state or the worst state has no impact in this study.

Moreover, as one might have noticed, our model is similar to the practical-in-used régime-switching-lognormal model (RSLN), where the distribution of the firm value's logarithm under each state i is further specified as normal distributions $N(\mu_i, \sigma_i)$; see Hamilton (1989) for details. Although it can be viewed as a parallel method of this kind of model, three contributions are noted in our framework. Since there is no restriction on the types of distributions in the model, it provides more freedom in model calibration and easy to use. First, we propose a Markov random walk model framework for bankruptcy prediction, in which it can capture both the differences in bankruptcy probabilities between firms with different ratings (exogenous), and between firms with the same rating but with different characteristics (endogenous). Second, we derive accurate numerical computation method to approximate the expected value and tail probabilities of the bankruptcy time. Third, unlike most RSLN models using unobservable macroeconomic stage as the underlying régime, our model uses observable rating process as the underlying Markov chain, which reduces the work of model calibration. Morever, this numerical approximation can also be used to evaluate the *power* of a credit rating system. Last but not least, even the RSLN model has been widely used both empirically and academically, its corresponding bankruptcy prediction has not yet been well established, the methods provided in this paper just fill in this gap. Henceforth, we think that our framework would be useful in bankruptcy prediction by considering both rating information and firm characteristics, and in the studying of efficiency for credit rating systems. Other important issues in credit risk such as credit derivatives and related topics after recent financial crisis can also be pursued under this model.

Last is the model calibration procedure. In order to build up a firm's model from observable data of stock markets and credit rating institutions, one may assume that the firm values' distributions, transition probability matrices and bankruptcy barriers are determined by the industry. The model parameters can then be calibrated as follows.

- First, one can replace the targeting firm's value by the book value of assets less than the book value of shareholders' equity, plus the market value of equity. This is a standard procedure that has been used by previous literatures, including Brockman and Turtle (2003). Same proxy should be taken throughout the next step.

- Second, for each given rating, one looks for similar firms as the desired company that stays in the specific rating, and collects the stock price of them. Then estimates the distribution parameters and barrier, which also gives the endogenous bankruptcy probability of the target firm. The computation of the implied parameters can be done through the similar process in Brockman and Turtle (2003). The existence of stable-rating firms is assured by the almost-diagonal transition probability matrix of the credit ratings. If, in rare cases, there is no firm in a specific rating, one may approximate the parameters in the no-existing state by the parameters from neighboring ratings; this could be done through linear interpolation or other methods. The ambiguity from this approximation would not affect much to the model since, as there is no firm in the specific rating, the transition probability going into this state would be small, and hence, this state's parameters would have a low weight of impact on the model as a whole.

- Third, one estimates the transition probability matrix of the credit ratings among the industry or similar corporations of the target company; this also determines the firm's exogenous bankruptcy probability. Usually, the rating institution's annual reports will provide the one-year transition probability matrices.

- Last, one needs to adjust the barriers to a single uniform barrier across states. If the maturity time was far enough, the underlying Markov chain, conditioned on not going into the bankruptcy state, converges to the invariant measure computed as the bankruptcy state is excluded. Therefore, we can set the desired single bankruptcy barrier as the weighted average of implied barriers with respect to the weights of each rating in the exogenous-bankruptcy-excluded invariant measure. This adjustment will change the bankruptcy probabilities in each state, in which it can be added back by adjusting the transition probability matrix with the bankruptcy state.

3 Bankruptcy Prediction

Up to now, we have developed a bankruptcy prediction model based on barrier-option framework as well as credit-rating information. In this section, we will further investigate bankruptcy time τ-related problems, in particular the approximation of $E_\nu(\tau)$ and $P_\nu\{\tau > t\}$ for $0 < t < T$, where ν denotes an initial distribution. Note that the τ-related terms can be used for bankruptcy prediction as well as pricing via equation (2). Moreover, since the exogenous bankruptcy time τ^{ex} depends only on the underlying Markov process, it is of interest to study the endogenous bankruptcy time τ^{en} for given non bankruptcy credit rating. Therefore, we will first approximate $E_\nu(\tau^{en})$ and $P_\nu\{\tau^{en} > t\}$, and then combine these results with the exogenous bankruptcy time τ^{ex} to have the approximations of $E_\nu(\tau)$ and $P_\nu\{\tau > t\}$. These two quantities can be regarded as proxies of the bankruptcy time τ, and have further applications in studying credit ratings and related topics, including the evaluation of the power for credit ratings systems.

The main goal of this section is to provide two types (credit-independent and credit-dependent) of approximations for the expected bankruptcy time $E_\nu(\tau^{en})$ and the tail probabilities $P_\nu\{\tau^{en} > t\}$ of the bankruptcy time τ^{en}. The credit-independent approximation is relatively easy to compute, however it is irrelevant to the initial rating as shown in (19) below. The credit-dependent approximation, on the other hand, reflects the differences in implied bankruptcy probabilities from distinct ratings and provides more accurate approximation. There are two essential advantages of the second approximation. First, it is essential for the approximation depends on initial credit rating, since it is natural for one to have different bankruptcy probabilities in each given credit rating. Therefore, the approximation which reflects the differences will be more suitable for credit rating researches and related topics. Second, it is vital to have high accuracy for the valuation of credit related securities. For instance, for corporate debts, whether the targeting firm goes into bankruptcy before the maturity can dramatically change the value of these securities. Even for the bankruptcy occurs at the day before maturity, or the day after it, would determine whether the par value will be repaid or not. Hence, this results in completely different valuations of these debts. The differences will be even more significant for credit swaps, credit protections, and related derivative instruments.

To have approximations of τ^{en}, in the following, we will assume that S_t is a Markov random walk without exogenous bankruptcy. That is, $q_{iK} = 0$ for all $1 \le i < K$. Let J_n be an ergodic (aperiodic, irreducible and positive recurrent) Markov chain on

$\{1, \ldots, K-1\}$, with the transition matrix given by

$$Q = (q_{ij})_{i,j=1}^{K-1} = \begin{pmatrix} 1 - \sum_{j \neq 1}^{K-1} \epsilon_{1j} & \epsilon_{12} & \cdots & \epsilon_{1(K-1)} \\ \epsilon_{21} & 1 - \sum_{j \neq 2}^{K-1} \epsilon_{2j} & \cdots & \epsilon_{2(K-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon_{(K-1)1} & \epsilon_{(K-1)2} & \cdots & 1 - \sum_{j \neq K-1}^{K-1} \epsilon_{(K-1)j} \end{pmatrix}. \tag{10}$$

Note that, as one may see in the following sections, the credit-dependent approximations are based on the Wald's equation in Theorem 1 of Fuh and Lai (1998) and equation (2.9) in Theorem 2 of Fuh (2004), and these formulae involve the following quantities: [3]

$$\rho_- = E_{\pi_-}(S_{\tau_-}^2)/2E_{\pi_-}(S_{\tau_-}), \tag{11}$$

where

$$\tau_- = \min\{n : S_n < 0, n > 0\}, \tag{12}$$

and π_- is the stationary distribution of the Markov chain with the transition probability matrix given by

$$q_{ij}^- = P_i \left\{ J_{\tau_-} = j \right\} \quad \text{for } i, j = 1, \ldots, K-1. \tag{13}$$

We will show how to approximate ρ_-. Note that although the approximation is not easy in Markov random walks, the terms $E(S_{\tau_-})$ and $E(S_{\tau_-}^2)$ appeared in simple random walks have been well studied, and can be explicitly computed via the characteristic function $g(\lambda) = E(e^{i\lambda X_1})$ of X_1 [cf. Chapter 10, Siegmund (1985)].

$$E(S_{\tau_-}) = \frac{1}{\sqrt{2}} \exp \frac{1}{\pi} \int_0^\infty \frac{1}{\lambda} Im\big[\ln(1 - g(-\lambda))\big] d\lambda, \tag{14}$$

and

$$\frac{E(S_{\tau_-}^2)}{2E(S_{\tau_-})} = -\frac{1}{6} E(X_1^3) - \frac{1}{\pi} \int_0^\infty \frac{1}{\lambda} Re\big[\ln \frac{2(1 - g(-\lambda))}{\lambda^2}\big] d\lambda. \tag{15}$$

By making use of (14) and (15), Theorem 1 provides an approximation of ρ_-.

Theorem 1. *Let J_n be an ergodic Markov chain on a finite state space $\{1, \ldots, K-1\}$, with transition probability matrix (10), and invariant measure $\pi = \{\pi_1, \cdots, \pi_{K-1}\}$. Let S_n be a Markov random walk defined on J_n such that $E_\pi(|X_1|^3) < \infty$. Then,*

$$R_- := \frac{\sum_{i=1}^{K-1} \sum_{j=1}^{K-1} \pi_i \times \big(q_{ij} + \sup\{q_{kj} | k \neq j\} \times (E_{max}^S(\tau_-) - 1)\big) \times E_j^S(S_{\tau_-}^2)}{2 \sum_{i=1}^{K-1} \sum_{j=1}^{K-1} \pi_i \times (q_{ij} + \sup\{q_{kj} | k \neq j\} \times (E_{max}^S(\tau_-) - 1)) \times E_j^S(S_{\tau_-})} \approx \rho_-, \tag{16}$$

where:

$$E_j^S(S_{\tau_-}) = E_j(S_{\tau_-} | \forall n \in \mathbb{N}, J_n = j), \tag{17}$$

and:

$$E_{max}^S(\tau_-) = \max\{E_j^S(\tau_-) : 1 \leq j \leq K-1\} \tag{18}$$

Furthermore, $|R_- - \rho_-| \to 0$ as $\sup_{1 \leq i,j \leq K-1, i \neq j} \epsilon_{ij} \to 0$.

[3] Notice that Fuh and Lai (1998) and Fuh (2004) study the stopping time $\tau(c) := \min\{n : S_n > c\}$, and hence consider $\tau_+ = \min\{n : S_n > 0\}$. We consider the descending ladder time to match our framework.

The main concept behind Theorem 1 is that, since the transition probabilities matrix is nearly diagonal, the credit rating process would mainly stays in a certain rating. Thus, we could approximate the Markov process by corresponding simple random walk, and hence, could estimate the desired ρ_- through (14) and (15). Further *off-diagonal adjustments* are done to take care of the rare probabilities that the rating changes, which is a crucial adjustment since when the barrier is not close to the initial firm value, the chance of going into another state is not ignorable, and thus need to take into considerations.

In the remaining of this section, we will apply Theorem 1 to approximate the endogenous bankruptcy time τ^{en}. Section 3.1 approximates the expected bankruptcy time τ^{en}. Section 3.2 estimates the tail probabilities of the bankruptcy time τ^{en}. In Section 3.3, we combine the results of the endogenous bankruptcy with the exogenous bankruptcy to derive an approximation of the combining bankruptcy. The proof of Theorem 1 is given in Appendix.

3.1 Approximations of $E_\nu(\tau^{en})$

For any given initial distribution (distribution of initial rating) ν, a simple way to estimate the expected bankruptcy time $E_\nu(\tau^{en})$ is via the following equation

$$E_\nu(\tau^{en}) \approx E_\pi(\tau^{en}) \approx \frac{H}{\mu_\pi}, \tag{19}$$

where $\mu_\pi = E_\pi(X_1) < 0$. If $|H|$ is large enough, the above credit-independent approximation can be accurate.

Note that approximation (19) contains no information about the initial distribution, which means the credit-independent approximation is irrelevant to the initial rating. However, to study the credit rating related topics, it is better to have an approximation that can reflect the differences in implied bankruptcy probabilities due to distinct initial ratings. This motivates us to propose a credit-dependent approximation. The main tool of this approximation is based on Wald's equation for Markov random walks in Fuh and Lai (1998). We include here for completeness.

Proposition 1. *Let $\{J_n, n \geq 0\}$ be an ergodic Markov chain on a finite state space $\{1, \ldots, K-1\}$ with stationary distribution π. Assume $E_\pi(|X_1|^3) < \infty$, $E_\pi(X_1) = \mu_\pi < 0$ and $S_0 = 0$. Then, for any given initial distribution ν of J_0, as $H \to -\infty$,*

$$\mu_\pi \times E_\nu(\tau^{en}) = H + \frac{E_{\pi_-}(S^2_{\tau_-})}{2E_{\pi_-}(S_{\tau_-})} + \sum_{j=1}^{K-1} \Delta(j)(\pi_j - \nu_j) + o(1), \tag{20}$$

where the vector $\Delta = \{\Delta(j)\}_{j=1}^{K-1}$ is given by the solution of the Poisson equation

$$(I - Q)\Delta = Q \begin{pmatrix} E(X_1|J_1 = 1) - \mu_\pi \\ E(X_1|J_1 = 2) - \mu_\pi \\ \vdots \\ E(X_1|J_1 = K-1) - \mu_\pi \end{pmatrix}. \tag{21}$$

Note that although the solutions of (21) are not unique, the differences $\Delta_i - \Delta_j$ are unique determined for $i, j = 1, \ldots, K-1$. Henceforth, the Wald's equation is well-defined. Combining Wald's equation (20) with Theorem 1, we have the credit-dependent[4] approximation for the expected bankruptcy time.

Theorem 2. *Assume the conditions in Proposition 1 hold. Then, as $H \to -\infty$ and $\epsilon_{ij} \to 0$ for all $1 \le i, j \le K-1$ and $i \ne j$,*

$$\mu_\pi \times E_\nu(\tau^{en}) = H + R_- + \sum_{j=1}^{K-1} \Delta(j)(\pi_j - \nu_j) + o(1), \tag{22}$$

where R_- is defined in Theorem 1.

The approximation (22) will be more accurate when $|H|$ is large, which means when the firm value is far from the bankruptcy barrier and has a negative mean. This is the case of a large firm that is decaying, which is just the situation that one would fairly care about the expected bankruptcy time when holding the firm's corporate bond, credit bankruptcy swaps and others.

Note that all terms in the credit-dependent approximation (22) can be explicitly computed via (14) and (15). Moreover, as the last terms in (22) depends on the initial state, the approximation does differs when the initial credit rating changes. Therefore, the credit-dependent approximation gives a better approximation in credit rating researches and related topics.

3.2 Approximate Tail Probabilities of τ^{en}

To approximate the tail probabilities of τ^{en}, we need the following result first. The reader is referred to Theorem 2 of Fuh (2004) for details. Denote $\sigma^2 = \lim_{n\to\infty} n^{-1} E_\nu((S_n - n\mu_\pi)^2)$, and $\kappa = \lim_{n\to\infty} n^{-1} E_\nu((S_n - n\mu_\pi)^3)$.

Proposition 2. *Let $\{J_n, n \ge 0\}$ be an ergodic Markov chain on a finite state space $\{1, \ldots, K-1\}$ with stationary distribution π. Assume $S_0 = 0$, $\mu_\pi = 0$, $\sigma = 1$, and $|\kappa| < \infty$. Denote $H = \xi\sqrt{t}$ for some $\xi < 0$. If there exists $C > 0$ such that $\inf_i P\{X_1 \le -C | J_1 = i\} > 0$. Then as $t \to \infty$,*

$$P_\pi\{\tau^{en} < t, S_t > H\} = \Phi\left(\frac{-\kappa/3 + H + 2\rho_-}{\sqrt{t + \kappa H/3}}\right) + o(\frac{1}{\sqrt{t}}). \tag{23}$$

Note that Proposition 2 holds for a general distribution after standardization i.e., $\mu_\pi = 0$, $\sigma = 1$. Furthermore, if X_1 is symmetric under each state, and the discretization period is small enough, then by reflection principle, we have

Theorem 3. *Under the conditions as Proposition 2. Assume X_1 is symmetric under each state, then*

$$\begin{aligned} P_\pi\{\tau^{en} > t\} &= 1 - P_\pi\{\tau^{en} < t\} \approx 1 - 2P_\pi\{\tau^{en} < t, S_t > H\} \\ &= 1 - 2\Phi\left(\frac{-\kappa/3 + H + 2\rho_-}{\sqrt{t + \kappa H/3}}\right) + o(\frac{1}{\sqrt{t}}). \end{aligned} \tag{24}$$

[4] Mathematically speaking, the credit-independent approximation for this expected value is also known as the first-order approximation, while the credit-dependent one is also known as the second-order approximation with error term of $o(1)$.

The constant term ρ_- in (24) can be estimated via R_- in Theorem 1. Equation (24) gives the credit-independent approximation of the tail probabilities, which is independent of the initial rating. To have a more accurate approximation, we need to incorporate the initial state information. Since the Markov chain is ergodic, the convergence of Q^n, nth convolution of Q, to the stationary distribution π is geometric fast, as $n \to \infty$. Therefore, one can choose a sufficient large N as a "cut-off point", such that the distribution of $\{J_n\}_{n=N+1}^{\infty}$ can be approximated by the stationary distribution; while the distribution of $\{J_n\}_{n=1}^{N}$ is govern by the initial state. Then one can replace $\{X_n\}_{n=1}^{N}$ with the distribution under the initial state. That is,

$$P_i\left\{\tau^{en} > t\right\} \approx P_\pi\left\{\tau^{en}(H - N \times E(X_1|J_1 = i)) > t - N\right\}, \tag{25}$$

for $1 << N < t$.

A more accurate approximation can be done by letting the drift-term decays exponentially from the initial state to the invariant measure, that is,

$$P_i\left\{\tau^{en} > t\right\} \approx P_\pi\left\{\tau^{en}\left(H - N \times \mu_\pi - (E(X_1|J_1 = i) - \mu_\pi) \times \sum_{n=1}^{N} q_{ii}^n\right) > t - N\right\}. \tag{26}$$

Here we call (26) as the credit-dependent approximation of the tail probabilities. More accurate approximation with $o(1)$ error term along with the proof is an open problem.

3.3 Approximations for the Bankruptcy Time τ and the Evaluation of Rating System's Performance

Now, we combine our previous results on endogenous bankruptcy with existed exogenous bankruptcy to have the bankruptcy time approximation. Let P^{en} be the probability with endogenous bankruptcy only. Then, a simple calculation leads to an estimation of the tail probabilities of τ

$$\begin{aligned} & P_\nu\{\tau > t\} = P_\nu\{\tau^{ex} > t, \tau^{en} > t\} = P_\nu\{\tau^{en} > t|\tau^{ex} > t\} \times P_\nu\{\tau^{ex} > t\} \\ = \; & P_\nu\{\tau^{en} > t|\text{The exogenous bankruptcy is excluded}\} \times P_\nu\{\tau^{ex} > t\} \\ = \; & P_\nu^{en}\{\tau^{en} > t\} \times P_\nu\{\tau^{ex} > t\}. \end{aligned} \tag{27}$$

Note that the first part of equation (27) can be approximated by (24) after adjusting the transition probability matrix conditioned on not in the bankruptcy state, and the second part can be computed directly through the transition probability matrix.

The approximation of $E_\nu(\tau)$ can be done by summing up equation (27), which leads to:

$$E_\nu(\tau) = E_\nu^{en}(\tau^{en}) - \sum_{t=1}^{\infty} P_\nu^{en}\{\tau^{en} \geq t\} P_\nu(\tau^{ex} < t). \tag{28}$$

Note that all terms related to endogenous bankruptcy could be computed through Section 3.1 and 3.2, while all terms related to exogenous bankruptcy could be directly calculated. The infinite summation could be done numerically. Notice that this gives (28) gives an upper bound of the expected bankruptcy time, and so does the expected exogenous bankruptcy time $E_\nu(\tau^{ex})$; thus, one can use the minimum of these to form an even better approximation of the expected bankruptcy time, which would be demonstrated in Section 4.3 and 4.4, where this minimum would be denoted as the *least-of-the-three* approximation.

These results provide us a method in studying rating systems' performance as we have stated in the introduction. By simulation, one could examine the proportion of exogenous bankruptcies among all bankruptcies. The larger the proportion, the stronger the rating information could determine the bankruptcy probabilities. Alternatively, one could use the following quantity, which we define as the *power* of the rating system,to examine the performance:

$$\text{Power} := \frac{E_\nu(\tau)}{E_\nu(\tau^{ex})} \tag{29}$$

Notice that this $0 < \text{power} \leq 1$ and, as the bankruptcy be determined more exogeneously, the power would be more close to 1. So, the larger the power is, the larger the proportion of exogenous bankruptcy is, and so the better the rating system performs.

4 Numerical Results

To examine the performance of our proposed approximations and application, we provides four numerical examples in this section, where the first three illustrations using given parameters to see their performance under the theoretical setting, and the last one using real to double confirm them. The first two examples are designed to examine our endogenous bankruptcy estimations provided in Sections 3.1 and 3.2, and the later two are for the combining bankruptcy estimations derived in Section 3.3.

More precisely, Section 4.1 considers the case of endogenous-bankruptcy-only with 3 states and given transition probability matrix, in which it is simple to compute and can be used for a simplified version of estimating the real cases with more states and complexity. Section 4.2 studies the case of endogenous-bankruptcy-only with 7 states using given transition matrix, which is closer to the real case in the number of states. Section 4.3 is a complete 8-state case, still with given matrix, with both endogenous and exogenous bankruptcies that allows us to examine the approximations when both bankruptcy barrier and bankruptcy state exist, and to demonstrate our application provided in Section 3.3. Last, Section 4.4 is similar to Section 4.3, but using historical Standard and Pool's credit-rating transition probability matrix of the year 2008, which additionally show that our model is able to capture the miss-ordering phenomenon of credit rating, that is a better rating but with higher bankruptcy probability.

The given parameters are chosen to approximate the empirical 1-year parameters, while the last illustrations maintain all parameters except the transition matrix in order to see the effect on the rating transition alone. These illustrations are numerical examples, not empirical studies, which is simply designed to demonstrate the function of our models, not to show any empirical phenomenon. We believe that these examples could lead to a new examining procedure in the study of rating systems.

4.1 Endogenous-Bankruptcy-Only: 3-state with Given Parameters

For simplicity, throughout the rest of this section, we will assume $X_1|J_j \backsim N(\mu_j, \sigma_j)$, the normality of X_1 in each state. Furthermore, to translate the yearly transition probability matrix into daily setting to be consistent with the firm value process's frequency, we will do the adjustment based on 220 working days per year throughout the rest of this section. That is,

$$Q_{adj} = \exp(\ln Q/220) \tag{30}$$

The first example deals with a 3-state transition probability matrix without the bankruptcy state. The reason for doing so is that, as we have mentioned in Section 2, the empirical credit ratings transition probability matrix is almost diagonal. Therefore, for a corporate security with short maturity, the firm's credit rating will barely move far away from the initial state, and most likely to stay in the current rating or the rating right above/below. Hence a 3-state case with the matrix centered at the initial state will be a good approximation for a real 7-state case when the maturity is not large.

Under this setting, the invariant distribution is given by

$$\begin{aligned}
\pi_1 &= \frac{(q_{21}+q_{23}+q_{32})q_{31}-(q_{31}-q_{21})q_{32}}{(q_{12}+q_{13}+q_{23})(q_{21}+q_{23}+q_{32})-(q_{31}-q_{21})(q_{32}-q_{12})},\\
\pi_2 &= \frac{-(q_{32}-q_{12})q_{31}+(q_{12}+q_{13}+q_{23})q_{32}}{(q_{12}+q_{13}+q_{23})(q_{21}+q_{23}+q_{32})-(q_{31}-q_{21})(q_{32}-q_{12})},\\
\pi_3 &= 1-\pi_1-\pi_2,
\end{aligned}$$

and the δ_j terms, as stated in Fuh and Pang (2009), are given by

$$\begin{aligned}
&\begin{pmatrix}\delta_1-\delta_2\\ \delta_2-\delta_3\\ \delta_3-\delta_1\end{pmatrix} \qquad (31)\\
=&\begin{pmatrix}-1 & -\frac{q_{22}(q_{31}+q_{32})+q_{23}q_{32}}{q_{21}(q_{31}+q_{32})+q_{31}q_{23}} & -\frac{q_{23}(q_{31}+q_{32})+q_{23}q_{33}}{q_{21}(q_{31}+q_{32})+q_{31}q_{23}}\\ 0 & -\frac{q_{22}q_{31}-q_{21}q_{32}}{q_{21}(q_{31}+q_{32})+q_{31}q_{23}} & -\frac{q_{23}q_{31}-q_{21}q_{33}}{q_{21}(q_{31}+q_{32})+q_{31}q_{23}}\\ 1 & -\frac{q_{32}(q_{21}+q_{22})+q_{23}q_{32}}{q_{21}(q_{31}+q_{32})+q_{31}q_{23}} & -\frac{q_{33}(q_{21}+q_{23})+q_{23}q_{32}}{q_{21}(q_{31}+q_{32})+q_{31}q_{23}}\end{pmatrix}\begin{pmatrix}E(X_1|J_1=1)-E_\pi(X_1)\\ E(X_1|J_1=2)-E_\pi(X_1)\\ E(X_1|J_1=3)-E_\pi(X_1)\end{pmatrix}.
\end{aligned}$$

As stated in Jarrow et al.(1997), a historical credit-rating transition probability matrix usually has its nonzero entries concentrated around the diagonal, and the probability of staying in the same rating is much bigger than the probability of moving to next (large or small) credit rating. To capture this phenomenon, we set the 1-year transition probability matrix as:

$$Q = \begin{pmatrix}0.89 & 0.1 & 0.01\\ 0.1 & 0.8 & 0.1\\ 0.01 & 0.1 & 0.89\end{pmatrix},$$

where $J_i = 1, 2, 3$ represent the Standard and Poor's rating AAA, AA and A respectively. X_n are then assumed to follow $N(\mu_i, \sigma_i)$ condition on each given $J_n = i$, where the parameters are chosen to represent a firm-value process under a specific credit rating, and are reported in Table 1[5]. For the barrier H, we follow the result in Brockman and

[5] More precisely, we select representative firms of AAA and CCC ratings respectively, calculate the log-return process of its stock price across the year 2008, and compute the daily mean and standard deviation of it as our proxy parameters. For other ratings, we use linear interpolation. For example, the Federal Bank(AAA) has a daily standard deviation of 0.0319, while Ford Motor Co.(CCC) has a standard deviation of 0.0482.

Turtle(2003), which is also reported in Table 1[6].

i	μ_i	σ_i	**S&P Rating**	**H**
1	-0.0006	0.0319	AAA	-9.2672
2	-0.0007	0.0346	AA	
3	-0.0008	0.0373	A	

Table 1: Parameters for 3-State Simulations(Daily)

We run a Monte Carlo simulation with $1,000$ paths as the proxy of the real expected value, and use it as our benchmark to compare with our approximations. The results are reported in Table 2. Notice that although the credit-independent approximation performs not bad, it is irrelevant to the initial rating. The credit-dependent approximation, on the other hand, not only provides more accurate results for any given initial state, but also has different values for different initial credit ratings, which is crucial in the study of credit rating system. As a result, it is better to use the credit-dependent approximation when one needs to apply this model and study the credit-rating related topics.

Initial State	**Simulated $E_\nu(\tau^{en})$** (Days. 1,000 Paths)	**Dependent Approx.**	**Independent Approx.**
AAA	13619	13514	13239
AA	13482	13268	13239
A	12947	13022	13239

Table 2: Results of 3-State Illustration for $E_\nu(\tau^{en})$

Last, Table 3 reports simulated tail probabilities and their estimations. The credit-independent approximation uses equation (24), and the credit-dependent approximation uses equation (26) in which it divides the process as the initial state-part and the invariant-measure part. The The cut-off point N was chosen to be 100. Note that the results are similar to the one of expected bankruptcy times, where the credit-independent approximations perform not bad, but the credit-dependent estimations show higher accuracy and reflect the differences of distinct states.

4.2 Endogenous-Bankruptcy-Only: 7-state with Given Parameters

Example 2 provides another numerical illustration of how to estimate with 7 states, which the number of states is much closer to many rating systems in practice, such as the Standard and Poor's. We set the 1-year rating transition matrix in the way similar to the 3-state case, which is presented in Table 4, which also shows the parameters of $N(\mu_j, \sigma_j)$

[6] Brockman and Turtle(2003) report an average firm-value of $6,662.53$ and average bankruptcy barrier of 0.6920, which is equivalent of a choice of $H = \ln 6662.53/0.6920 = -9.2672$ in our model.

t	15 years			20 years		
State	AAA	AA	A	AAA	AA	A
Simulated Prob.(1,000 paths)	0.9890	0.9740	0.9680	0.9600	0.9340	0.9280
Dependent Approx.(N=100)	0.9874	0.9823	0.9756	0.9509	0.9365	0.9191
Inependent Approx.	0.9797	0.9797	0.9797	0.9306	0.9306	0.9306

Table 3: Results of 3-State Illustration for $P_\nu\{\tau^{en} > t\}$

	Probability To							Daily Parameters	
Current State	AAA	AA	A	BBB	BB	B	CCC/C	μ	σ
AAA	0.8900	0.1000	0.0100	0.0000	0.0000	0.0000	0.0000	-0.0006	0.0319
AA	0.1000	0.7900	0.1000	0.0100	0.0000	0.0000	0.0000	-0.0007	0.0346
A	0.0100	0.1000	0.7800	0.1000	0.0100	0.0000	0.0000	-0.0008	0.0373
BBB	0.0000	0.0100	0.1000	0.7800	0.1000	0.0100	0.0000	-0.0009	0.0401
BB	0.0000	0.0010	0.0100	0.1000	0.7800	0.1000	0.0100	-0.0010	0.0428
B	0.0000	0.0000	0.0000	0.0100	0.1000	0.7900	0.1000	-0.0011	0.0455
CCC/C	0.0000	0.0000	0.0000	0.0000	0.0100	0.1000	0.8900	-0.0012	0.0482

Table 4: Parameters for the 7-State Numerical Illustration

State	AAA	AA	A	BBB	BB	B	CCC/C
Simulation (Days)	12213	11654	11348	10668	9800	9175	9118
Dependent Approx.	12234	11860	11161	10331	9506	8828	8401
Independent Approx.	10348	10348	10348	10348	10348	10348	10348

Table 5: Results of 7-State Illustration for $E(\tau^{en})$

for X_1 given $J_1 = j$.

The barrier H is assumed to be the same as in Section 4.1, where the benchmark is still set to be the result of Monte Carlo simulation with 1000 paths. The results are reported in Table 5, which, in consistent with the 3-state case, demonstrate the differences between credit-independent and -dependent approximations' performances.

Last, Table 6 reports the simulated tail probabilities and their estimations under 7-state case. The simulated paths are the same as in Table 5, and the cut-off point N used in the credit-dependent approximation is 100. For simplicity, we only report the results of the first, the middle, and the last ratings. The results are generally in consist with Section 4.1. Notice that the the credit-dependent outperforms the other improvement only occurs in the CCC/C case.

4.3 Combining Bankruptcy and Rating System Evaluations: 8-state with Given Parameters

Example 3 provide the numerical illustration that combines exogenous and endogenous bankruptcies by having bankruptcy barrier and bankruptcy state simultaneously. The transition probabilities of going into the bankruptcy state are set from 0 to 0.32, where the probability from the next rating is two times as the one from the last rating. The rest of the transition probabilities are set so that the conditional transition probability matrix of not going into the bankruptcy state is the same as the matrix used in Section 4.2. All other parameters remain the same. We present these parameters in Table 7.

t	15 years			20 years		
State	AAA	BBB	CCC/C	AAA	BBB	CCC/C
Simulation Prob.	0.9720	0.9140	0.7960	0.9180	0.8190	0.6900
Dependent Approx.(N=100)	0.9291	0.8822	0.8148	0.8187	0.7362	0.6319
Independent Approx.	0.8726	0.8726	0.8726	0.7225	0.7225	0.7225

Table 6: Results of 7-State Illustration for $P_\nu\{\tau^{ex} > t\}$

Current State	**Probability To** AAA	AA	A	BBB	BB	B	CCC/C	Bankruptcy	**Parameters** μ	σ
AAA	0.8900	0.1000	0.0100	0.0000	0.0000	0.0000	0.0000	0.0000	-0.0006	0.0319
AA	0.0990	0.7821	0.0990	0.0099	0.0000	0.0000	0.0000	0.0100	-0.0007	0.0346
A	0.0098	0.0980	0.7644	0.0980	0.0098	0.0000	0.0000	0.0200	-0.0008	0.0373
BBB	0.0000	0.0096	0.0960	0.7488	0.0960	0.0096	0.0000	0.0400	-0.0009	0.0401
BB	0.0000	0.0000	0.0092	0.0920	0.7176	0.0920	0.0092	0.0800	-0.0010	0.0428
B	0.0000	0.0000	0.0000	0.0084	0.0840	0.6636	0.0840	0.1600	-0.0011	0.0455
CCC/C	0.0000	0.0000	0.0000	0.0000	0.0068	0.0680	0.6052	0.3200	-0.0012	0.0482
Bankruptcy	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000		

Table 7: Parameters for the 8-State Numerical Illustration with Given Parameters

EXPECTATION $E_\nu(\tau)$

State	AAA	AA	A	BBB	BB	B	CCC/C
Simulation(1,000 paths)	9040	7596	5994	4363	2822	1564	762
Least-of-the-Three	8965	8369	7551	6099	3659	1878	848
$E_\nu(\tau^{ex})$	12874	11104	8763	6099	3659	1878	848
Dependent Approx.(N=100)	11466	10307	8638	6597	4537	2817	1900
Independent Approx.	8965	8369	7551	6492	5341	4325	3612

TAIL PROBABILITIES $P_\nu\{\tau > t\}$

t	15 years			20 years		
State	AAA	BBB	CCC/C	AAA	BBB	CCC/C
Simulation(1,000 paths)	0.9160	0.4840	0.0340	0.8550	0.3770	0.0180
Dependent Approx.(N=100)	0.8993	0.4756	0.0328	0.8153	0.3648	0.0214
Independent Approx.	0.8931	0.4744	0.0330	0.7872	0.3648	0.0200

Table 8: Results of 8-State Illustration with Given Parameters

We apply the method derived in Section 3.3 to approximate both $E_\nu(\tau)$ and $P_\nu\{\tau < t\}$. The number of paths is $1,000$, the s value in computing for expectation is set to be $0.5 \times E_\nu(\tau^{ex})$, and the cut-off point N for the credit-dependent approximation is 100. The results were presented in Table 8, in which, we only report the tail probabilities and their approximations of the first, the middle, and the last ratings for simplicity. For the expected bankruptcy time, notice that the expected exogenous bankruptcy time and both of the approximations are upperbounds for $E_\nu(\tau)$, so we may choose the least of the three to form a good estimation of $E_\nu(\tau)$, which is shown in Table 8. And for the tail probabilit, the results are generally in consistent with the previous illustrations, where the major disimprovement occurs in the CCC/C rating, which is the least diagonal state among all.

Last, we use this illustration to present how to examine the performance of a rating system. The first method is to compute the ratio of exogenous bankruptcies among all bankruptcies by Monte Carlo simulation with 1,000 paths. The results are shown in Table 9. Notice that the ratio is ascending from the best rating to the worst rating, which suggest that given the CCC/C rating, one could know much more information about the firm's bankruptcy probability then given other rating.

We also present the power of each rating in Table 9, which is defined in (**??**), where the least-of-the-three approximations are applied. Notice that the second method is explicitly

State	AAA	AA	A	BBB	BB	B	CCC/C
Exogeneous Bankruptcy Ratio	0.5420	0.6320	0.7170	0.8240	0.9150	0.9580	0.9880
Power(By Least-of-the-Three)	0.6964	0.7537	0.8617	1.0000	1.0000	1.0000	1.0000

Table 9: Results of Exogenous Bankruptcy Ratio and Powers with 8 States, Given Parameters

	Probability To								**Parameters**	
Current State	AAA	AA	A	BBB	BB	B	CCC/C	Bankruptcy	μ	σ
AAA	0.8710	0.0645	0.0323	0.0000	0.0000	0.0101	0.0202	0.0000	-0.0006	0.0319
AA	0.0000	0.8087	0.1794	0.0059	0.0000	0.0000	0.0020	0.0040	-0.0007	0.0346
A	0.0000	0.0167	0.9227	0.0518	0.0047	0.0000	0.0000	0.0040	-0.0008	0.0373
BBB	0.0000	0.0000	0.0274	0.9244	0.0382	0.0029	0.0021	0.0050	-0.0009	0.0401
BB	0.0000	0.0010	0.0000	0.0535	0.8365	0.0895	0.0113	0.0082	-0.0010	0.0428
B	0.0000	0.0000	0.0000	0.0016	0.0414	0.8231	0.0909	0.0430	-0.0011	0.0455
CCC/C	0.0000	0.0000	0.0000	0.0000	0.0000	0.1410	0.5257	0.3333	-0.0012	0.0482
Bankruptcy	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	1.0000		

Table 10: Parameters for the 8-State Numerical Illustration with Historical Parameters

computable, and captures the pattern of the exogenous bankruptcy ratio, so it would lead to the same conclusions. Thus, one could evaluate the rating system's performance by calculating the power instead of computing the ratio to avoid the simulations. A similar method could be used to evaluate the performance between different rating systems.

4.4 Combining Bankruptcy and Rating System Evaluations: 8-state with Historical Parameters

Lastly, Example 4 redo the 8-state case in Example 4, but replace the rating transition matrix with the historical data, Standard and Poor's 1-year transition matrix to be precise. We use this example to demonstrate the capability of our framework in dealing with the real data, and additionally show that our model is able to capture the miss-ordering phenomenon of credit rating, that is a better rating with a larger bankruptcy probability.

The parameters are shown in Table 10, with all parameters being the same as in Example 3 except the rating transition matrix. The results in simulations and approximations are shown in Table 11, with the number of paths been set as $1,000$, the s value in computing for expectation been set as $0.5 \times E_\nu(\tau^{ex})$, and the cut-off point N for the credit-dependent approximation been set as 100. The results were presented in Table 11, in which, we only report the results of the first, the middle, and the last ratings for simplicity. The results are generally in consistent with the previous illustrations, where the only disimprovement of the tail probability approximations occurs in the less diagonal CCC/C rating.

Notice that the expected bankruptcy time in AAA rating is smaller than the one in AA or A rating, which is because the transition probability from AAA to B and CCC/C is comparably larger than from the AA rating. This demonstrates that our model is capable in capturing the miss-ordering phenomenon, that is a better rating but with a larger bankruptcy probability, possibly because of some extreme condition, such as the financial crisis in the year 2008 which our parameters are set in.

EXPECTATION $E_\nu(\tau)$

State	AAA	AA	A	BBB	BB	B	CCC/C
Simulation(1,000 paths)	7889	8803	8235	7298	5034	2848	1245
Least-of-the-Three	7956	8413	8374	8036	6866	3938	1549
$E_\nu(\tau^{ex})$	13759	15528	14933	12383	7742	3938	1549
Dependent Approx.(N=100)	9614	10570	10266	9043	6866	4957	3614
Independent Approx.	7956	8413	8374	8036	6968	5559	4122

TAIL PROBABILITIES $P_\nu\{\tau > t\}$

t	15 yeas			20 years		
State	AAA	BBB	CCC/C	AAA	BBB	CCC/C
Simulation(1,000 paths)	0.8310	0.8390	0.1090	0.7520	0.7430	0.0830
Dependent Approx.(N=100)	0.8127	0.8328	0.1068	0.7281	0.7373	0.0757
Independent Approx.	0.7982	0.8241	0.1073	0.6769	0.6829	0.0703

Table 11: Results of 8-State Illustration with Historical Parameters

State	AAA	AA	A	BBB	BB	B	CCC/C
Exogeneous Bankruptcy Ratio	0.4900	0.3660	0.4060	0.4960	0.7080	0.8800	0.9580
Power(By Least-of-the-Three)	0.5782	0.5418	0.5608	0.6490	0.8869	1.0000	1.0000

Table 12: Results of Exogenous Bankruptcy Ratio and Powers with 8 States, Historical Parameters

Lastly, we redo the examination in Section 4.3 as shown in Table 12. Notice that the pattern of the power also matches the pattern of the ratio, which means that one could use the explicitly computable power instead of the simulated ratio to examine the rating systems. Moreover, notice that the examination also captures the miss-ordering phenomenon where the AAA rating has a higher ratio and power than the AA and A ratings. We believe that this illustration demonstrates that our framework could be applied to the real world data.

5 Conclusion

In this paper, three major parts are provided: a new model is formulated, two types of estimations are derived, and an application of examine the power of rating systems is proposed. The model can capture both the differences in bankruptcy probabilities between firms with different ratings, and between firms with the same rating but with different characteristics. The approximations are explicitly computable, where one is easier to calculate, and the other is more accurate and could reflect the differences due to different ratings. And the application could use to examine the proportion of bankruptcy that is determined by the credit rating system, which indicates how well we could know about bankruptcy probability of a firm when given the firm's rating information only, and thus could be used to evaluate the rating system's performance.

We believe that our works provides sufficient tools in studying bankruptcies related to rating transition, the performance of the rating systems, and other rating related topics. Bankruptcy correlation studies are also possible; one can use this model to link the bankruptcy correlations with the correlations of credit-rating changing, which can possibly be a consequence of changes in sovereign rating or macro conditions. Credit-related securities valuation and many other credit related issues arise after the recent financial crisis are still possible.

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Appendix. Proof of Theorem 1

The proof of Theorem 1, the estimation of $\rho_- = E_{\pi_-}(S^2_{\tau_-})/2E_{\pi_-}(S_{\tau_-})$, could be divided into three steps. First, we derive representations of the probabilities $P_i\{S_{\tau_-} \geq x\}$ and $P_i\{S^2_{\tau_-} \geq x\}$ for any given initial state $i = 1, \ldots, K-1$. Second, for each given initial state i, we estimate $E_i(S_{\tau_-})$ and $E_i(S^2_{\tau_-})$ by their corresponding simple random walks. Third, we estimate $E_{\pi_-}(S_{\tau_-})$ and $E_{\pi_-}(S^2_{\tau_-})$, and therefore provide the estimation of ρ_-.

A.1 Representation of Tail Probabilities for S_{τ_-}

We begin our method with the following representation of the joint probabilities of $S^\alpha_{\tau_-}$ and J_{τ_-} for $\alpha = 1, 2$. For simplicity, for the rest of this appendix, we would use the notations P_i^S and E_i^S to denote the probability and the expectation where the underlying Markov chain is conditioned on state i, which degenerates the process into a simple random walk. For example,

$$E_i^S(S_{\tau_-}) = E_i(S_{\tau_-} | \forall n \in \mathbb{N}, J_n = i). \tag{32}$$

The notations P_ν and E_ν would always mean the probability and expectation with J_0 having the initial distribution ν, and for $i \in \{1, \cdots, K-1\}$, P_i and E_i would always mean the probability and expectation with $J_0 = i$. Moreover, S_0 is assumed to be 0 if not further specified. The transition probability matrix, on the other hand, would be given by:

$$Q = (q_{ij})_{i,j=1}^{K-1} = \begin{pmatrix} 1 - \sum_{j\neq 1}^{K-1} \epsilon_{1j} & \epsilon_{12} & \cdots & \epsilon_{1(K-1)} \\ \epsilon_{21} & 1 - \sum_{j\neq 2}^{K-1} \epsilon_{2j} & \cdots & \epsilon_{2(K-1)} \\ \vdots & \vdots & \vdots & \vdots \\ \epsilon_{(K-1)1} & \epsilon_{(K-1)2} & \cdots & 1 - \sum_{j\neq K-1}^{K-1} \epsilon_{(K-1)j} \end{pmatrix}, \tag{33}$$

where $0 \leq \epsilon_{ij} \leq 1$ and $\sum_{j\neq i}^{K-1} \epsilon_{ij} \leq 1$ for all $1 \leq i, j \leq K-1$ and $i \neq j$.

Lemma 1 (Representation of the Joint Probabilities). *For $\alpha = 1, 2$, $x \geq 0$, and $i, j \in \{1, \cdots, K-1\}$:*

$$\begin{aligned} P_i\left\{|S_{\tau_-}|^\alpha \geq x, J_{\tau_-} = j\right\} = &\sum_{\tau=1}^{\infty} q_{ij} q_{jj}^{\tau-1} P_i^S\left\{|S_\tau|^\alpha \geq x, \tau_- = \tau\right\} \\ &+ \sum_{m=1}^{\infty}\sum_{n_v=1}^{\infty}\sum_{k\neq j}^{K-1} q_{kj} q_{jj}^{\tau-1} \int_0^\infty P_i\left\{S_{n_v} = y, \tau_- > n_v, J_{n_v} = k\right\} \\ &\times P_j^S\left\{|S_m|^\alpha \geq (x^{\frac{1}{\alpha}} + y)^\alpha, \tau_- = m\right\} dy. \end{aligned} \tag{34}$$

Proof. We provide only the proof for $i = j = 1$ and $\alpha = 1$. The proof for the rest is pretty much the same. Denote:

$$N_v = \begin{cases} \max\left\{n_v \leq \tau_- : J_{n_v} \neq 1\right\} & \text{if } \exists n \leq \tau_- \text{ s.t. } J_n \neq 1 \\ 0 & \text{if } \nexists n \leq \tau_- \text{ s.t. } J_n \neq 1 \end{cases}$$

be the last index before τ_- that makes the state process NOT in state 1. Since $J_n = 1$ for all $N_v < n \le \tau_-$, the process $\{S_n\}$ beyond $n = N_v$ can be treated as a simple random walk that is "fixed" in state 1. Thus, we can divide the whole process $\{S_n\}_{n=1}^{\tau_-}$ into a "variant-state" part $\{S_n\}_{n=1}^{N_v}$ and a "fixed-state" part $\{S_n\}_{n=N_v+1}^{\tau_-}$. This is the key technique that we use to estimate the Markov random walk case via simple random walk cases.

The detail goes as follows. Notice that:

$$
\begin{aligned}
P_1\left\{|S_\tau| \ge x, J_\tau = 1, \tau_- = \tau\right\} = & \sum_{n_v=0}^{\tau-1} P_1\left\{S_\tau \le -x, J_\tau = 1, \tau_- = \tau, N_v = n_v\right\} \\
=& P_1\left\{S_\tau \le -x, J_\tau = 1, \tau_- = \tau, N_v = 0\right\} \\
& + \sum_{n_v=1}^{\tau-1}\sum_{k=2}^{K-1} P_1\left\{S_\tau \le -x, J_\tau = 1, \tau_- = \tau, N_v = n_v, J_{n_v} = k\right\} \\
=& A_0 + \sum_{n_v=1}^{\tau-1}\sum_{k=2}^{K} A_{n_v,k}.
\end{aligned} \tag{35}
$$

For A_0, since $N_v = 0$ indicates that the entire process stays in state 1, therefore,

$$
\begin{aligned}
A_0 =& P_1\left\{S_\tau \le -x, J_1 = J_2 = \cdots = J_\tau = 1, \tau_- = \tau\right\} \\
=& P_1\left\{J_1 = J_2 = \cdots = J_\tau = 1\right\} \times P_1\left\{S_\tau \le -x, \tau_- = \tau | J_1 = J_2 = \cdots = J_\tau = 1\right\} \\
=& q_{11}^\tau P_1^S\left\{S_\tau \le -x, \tau_- = \tau\right\}
\end{aligned} \tag{36}
$$

For $A_{n_v,j}$, divide the process into the "variant-state" part $\{S_n\}_{n=1}^{n_v}$ and the "fixed-state" part $\{S_n\}_{n=n_v+1}^{\tau_-}$. The variant-state part requires the Markov chain $\{J_n\}_{n=1}^{n_v}$ to end in state j; this probability could be calculated through the transition probability matrix. The fixed-state part, on the other hand, requires the Markov chain $\{J_n\}_{n=n_v+1}^{\tau_-}$ to stay in state 1, and could be treated as a simple random walk that starts from S_{n_v}, which must be positive since $\tau_- > n_v$. Namely,

$$
\begin{aligned}
A_{n_v,k} =& P_1\left\{S_\tau \le -x, \tau_- = \tau, J_{n_v} = k, J_{n_v+1} = \cdots = J_n = 1\right\} \\
=& P_1\left\{S_\tau \le -x, \tau_- = \tau, \tau_- > n_v, J_{n_v} = k, J_{n_v+1} = \cdots = J_\tau = 1\right\} \\
=& \int_0^\infty P_1\left\{S_\tau \le -x, S_{n_v} = y, \tau_- = \tau, \tau_- > n_v, J_{n_v} = k, J_{n_v+1} = \cdots = J_\tau = 1\right\} dy \\
=& \int_0^\infty P_1\left\{S_\tau \le -x, \tau_- = \tau | S_{n_v} = y, \tau_- > n_v, J_{n_v} = k, J_{n_v+1} = \cdots = J_\tau = 1\right\} \\
& \times P_1\left\{S_{n_v} = y, \tau_- > n_v | J_{n_v} = k, J_{n_v+1} = \cdots = J_\tau = 1\right\} \\
& \times P_1\left\{J_{n_v+1} = \cdots = J_\tau = 1 | J_{n_v} = k\right\} \times P_1\left\{J_{n_v} = k\right\} dy \\
=& P_1\left\{J_{n_v+1} = \cdots = J_\tau = 1 | J_{n_v} = k\right\} \\
& \times \int_0^\infty P_1\left\{S_n - S_{n_v} \le -x - y, \tau_- = \tau | S_{n_v} = y, \tau_- > n_v, J_{n_v} = k, J_{n_v<n\le\tau} = 1\right\} \\
& \times P_1\left\{S_{n_v} = y, \tau_- > n_v | J_{n_v} = k\right\} \times P_1\left\{J_{n_v} = k\right\} dy \\
=& q_{k1} q_{11}^{\tau-n_v-1} \int_0^\infty P_1^S\left\{S_{\tau-n_v} \le -x - y, \tau_- = \tau - n_v\right\} \\
& \times P_1\left\{S_{n_v} = y, \tau_- > n_v, J_{n_v} = k\right\} dy.
\end{aligned} \tag{37}
$$

The last equality comes from the fact that given $J_n = 1$ for all $i \in \mathbb{N}$, $S_n - S_{n_v}$ has the same distribution as S_{n-n_v}. Putting these back into equation (35), and we achieve:

$$\begin{aligned}(35) =& q_{11}^{\tau} P_1^S \{S_\tau \le -x, \tau_- = \tau\} \\ &+ \sum_{n_v=1}^{n-1} \sum_{k=2}^{K-1} q_{k1} q_{11}^{\tau-n_v-1} \int_0^\infty P_1^S \{S_{\tau-n_v} \le -x-y, \tau_- = \tau - n_v\} \\ &\times P_1 \{S_{n_v} = y, \tau_- > n_v, J_{n_v} = k\} \, dy.\end{aligned}$$

By summing the last equation with respect to τ, and make a change of variable $m = \tau - n_v$, we complete the proof.

$\square$

Summing up the equality in Lemma 1 with respect to j gives us a representation of $P\{|S_{\tau_-}|^\alpha \ge x\}$.

A.2 Approximation of S_{τ_-}'s Moments

Now, the approximation of $E(S_{\tau_-}^\alpha)$ is ready to be achieved through Lemma 1.

Lemma 2 (Approximations of S_{τ_-}'s Moments). *For all $i \in \{1, \cdots, K-1\}$ and $\alpha = 1, 2$,*

$$E_i(|S_{\tau_-}|^\alpha) \le \sum_{j=1}^{K-1} (q_{ij} + \sup\{P_i \{J_n \ne j, J_{n+1} = j\} : n \in \mathbb{N}\} \times E_i(\tau_- - 1)) \times E_j^S(|S_{\tau_-}|^\alpha), \tag{38}$$

where the difference between the two converges to zero as $\sup_{1 \le i,j \le K-1, i \ne j} \epsilon_{ij} \to 0$.

Proof. Similarly, we only provide the proof for $i = 1$ and $\alpha = 1$. Lemma 1 yields:

$$\begin{aligned}E_1(|S_{\tau_-}|) =& \int_0^\infty \sum_{j=1}^{K-1} \sum_{\tau=1}^{\infty} q_{1j} q_{jj}^{\tau-1} P_j^S \{|S_\tau| \ge x, \tau_- = \tau\} \, dx \\ &+ \int_0^\infty \sum_{j=1}^{K-1} \sum_{m=1}^{\infty} \sum_{n_v=1}^{\infty} \sum_{k \ne j}^{K-1} q_{kj} q_{jj}^{m-1} \int_0^\infty P_1 \{S_{n_v} = y, \tau_- > n_v, J_{n_v} = k\} \\ &\times P_j^S \{|S_m| \ge x + y, \tau_- = m\} \, dy dx.\end{aligned} \tag{39}$$

For the first part, notice that since $q_{jj} \le 1$,

$$\begin{aligned}&\int_0^\infty \sum_{j=1}^{K-1} \sum_{\tau=1}^{\infty} q_{1j} q_{jj}^{\tau-1} P_j^S \{|S_\tau| \ge x, \tau_- = \tau\} \, dx \\ \le& \sum_{j=1}^{K-1} q_{1j} \int_0^\infty \sum_{\tau=1}^{\infty} P_j^S \{|S_\tau| \ge x, \tau_- = \tau\} \, dx \\ =& \sum_{j=1}^{K} q_{1j} E_j^S(|S_{\tau_-}|).\end{aligned} \tag{40}$$

For the second part, notice that since $y \geq 0$ and $q_{jj} \leq 1$:

$$\begin{aligned}
&\int_0^\infty \sum_{m=1}^\infty q_{jj}^{m-1} P_j^S \left\{ |S_m| \geq x + y, \tau_- = m \right\} dx \\
\leq &\int_0^\infty \sum_{m=1}^\infty P_j^S \left\{ |S_m| \geq x, \tau_- = m \right\} dx \\
= &E_j^S(|S_{\tau_-}|). \qquad (41)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{Second Part } &\leq \sum_{j=1}^{K-1} E_j^S(|S_{\tau_-}|) \times \sum_{n_v=1}^\infty \sum_{k\neq j}^{K-1} q_{kj} \int_0^\infty P_1 \left\{ S_{n_v} = y, \tau_- > n_v, J_{n_v} = k \right\} dy \\
&= \sum_{j=1}^{K-1} E_j^S(|S_{\tau_-}|) \times \sum_{n_v=1}^\infty P_1 \left\{ \tau_- > n_v, J_{n_v} \neq j, J_{n_v+1} = j \right\} dy \\
&= \sum_{j=1}^{K-1} E_j^S(|S_{\tau_-}|) \times \sum_{n_v=1}^\infty P_1 \left\{ \tau_- > n_v | J_{n_v} \neq j \right\} P_1 \left\{ J_{n_v} \neq j, J_{n_v+1} = j \right\} \\
&\leq \sum_{j=1}^{K-1} \sup\{ P_1 \left\{ J_n \neq j, J_{n+1} = j \right\} : n \in \mathbb{N} \} \times E_1(\tau_- - 1) \times E_j^S(|S_{\tau_-}|). \qquad (42)
\end{aligned}$$

Plug these back into (39), and we complete the proof. Notice that the equality holds when $q_{ii} = 1$, that is, when the Markov random walk degenerates to a simple random walk. □

For computational concern, one can use a more loosen upper-bound:

Corollary 1. *Let* $E_{max}^S(\tau_-) = \max\{E_l^S(\tau_-) : 1 \leq l \leq K-1\}$. *Then,*

$$E_i(|S_{\tau_-}|^\alpha) \leq \sum_{j=1}^{K-1} \big(q_{ij} + \sup\{q_{kj} | k \neq j\} \times (E_{max}^S(\tau_-) - 1) \big) \times E_j^S(|S_{\tau_-}|^\alpha), \qquad (43)$$

where the difference between the two converges to zero as $\sup_{1\leq i,j \leq K-1, i\neq j} \epsilon_{ij} \to 0$.

The advantage of this corollary 1 is because that $E_i^S(\tau_+)$ could be computed explicitly via the Wald's equation of simple random walks:

$$E_i^S(X_1) \times E_i^S(\tau_-) = E_i^S(S_{\tau_-}), \qquad (44)$$

and could be further computed via (14).

A.3 Estimating $\rho_- = E_{\pi_-}(S_{\tau_-}^2)/2E_{\pi_-}(S_{\tau_-})$

Last, we apply Corollary 1 to estimate ρ_-. Notice that:

Lemma 3. $||\pi_- - \pi|| \to 0$ *as* $\epsilon_{ij} \to 0$ *for all* $1 \leq i, j \leq K-1$ *and* $i \neq j$, *where* $|| \bullet ||$ *is the usual inner-product norm.*

Proof. Notice that:

$$q_{ij}^{-} = P_i \left\{ J_{\tau_-} = j \right\} = P_i \left\{ S_{\tau_-} \leq 0, J_{\tau_-} = j \right\}. \tag{45}$$

So by Lemma 1, one can easily show that $q_{ij}^{-} \to q_{ij}$ as $\epsilon_{ij} \to 0$ for all $1 \leq i, j \leq K-1$ and $i \neq j$. This completes the proof. $\square$

Thus, by taking the weighted average of the results in Corollary 1, it is evident that, as $\epsilon_{ij} \to 0$ for all $1 \leq i, j \leq K-1$ and $i \neq j$,

$$\begin{aligned} R_- :=& \frac{\sum_{i=1}^{K-1} \sum_{j=1}^{K-1} \pi_i \times \left(q_{ij} + \sup\{q_{kj} | k \neq j\} \times (E_{max}^S(\tau_-) - 1) \right) \times E_j^S(S_{\tau_-}^2)}{2 \sum_{i=1}^{K-1} \sum_{j=1}^{K-1} \pi_i \times (q_{ij} + \sup\{q_{kj} | k \neq j\} \times (E_{max}^S(\tau_-) - 1)) \times E_j^S(S_{\tau_-})} \\ & \to \frac{E_{\pi_-}(S_{\tau_-}^2)}{2 E_{\pi_-}(S_{\tau_-})} = \rho_- \end{aligned} \tag{46}$$

which completes the proof of Theorem 1.