Supplement Chapter 5

Alternative Theories for Deriving the Optimal Hedge Ratio

5.1 Introduction

The basic concept of hedging is to combine investments in the spot market and futures market to form a portfolio that will eliminate (or reduce) fluctuations in its value. Specifically, consider a portfolio consisting of C_s units of a long spot position and C_f units of a short futures position.[1] Let S_t and F_t denote the spot and futures prices at time *t*, respectively. Since the futures contracts are used to reduce the fluctuations in spot positions, the resulting portfolio is known as the hedged portfolio. The return on the hedged portfolio, R_h, is given by:

$$R_h = \frac{C_s S_t R_s - C_f F_t R_f}{C_s S_t} = R_s - hR_f \,, \tag{5.1a}$$

where $h = \frac{C_f F_t}{C_s S_t}$ is the so-called hedge ratio, and $R_s = \frac{S_{t+1} - S_t}{S_t}$ and $R_f = \frac{F_{t+1} - F_t}{F_t}$ are so-called one-period returns on the spot and futures positions, respectively. Sometimes, the hedge ratio is discussed in terms of price changes (profits) instead of returns. In this case the profit on the hedged portfolio, ΔV_H, and the hedge ratio, H, are respectively given by:

$$\Delta V_H = C_s \Delta S_t - C_f \Delta F_t \quad \text{and} \quad H = \frac{C_f}{C_s} \,, \tag{5.1b}$$

where $\Delta S_t = S_{t+1} - S_t$ and $\Delta F_t = F_{t+1} - F_t$.

The main objective of hedging is to choose the optimal hedge ratio (either *h* or *H*). As mentioned above, the optimal hedge ratio will depend on a particular objective function to be optimized. Furthermore, the hedge ratio can be static or dynamic. In subsections A and B, we will discuss the static hedge ratio and then the dynamic hedge ratio.

It is important to note that in the above setup, the cash position is assumed to be fixed and we only look for the optimum futures position. Most of the hedging literature assumes that the cash position is fixed, a setup that is suitable for financial futures. However, when we are dealing with commodity futures, the initial cash position becomes an important decision variable that is tied to the production decision. One such setup considered by Lence (1995 and 1996) will be discussed in subsection *C*.

5.2 Static Case

We consider here that the hedge ratio is static if it remains the same over time. The static hedge ratios reviewed in this paper can be divided into eight categories, as shown in Table 5.1. We will discuss each of them in the paper.

Table 5.1: A List of Different Static Hedge Ratios

Hedge Ratio	Objective Function
• Minimum-Variance (MV) Hedge Ratio	Minimize variance of R_h
• Optimum Mean-Variance Hedge Ratio	Maximize $E(R_h) - \frac{A}{2} Var(R_h)$
• Sharpe Hedge Ratio	Maximize $\frac{E(R_h) - R_F}{\sqrt{Var(R_h)}}$
• Maximum Expected Utility Hedge Ratio	Maximize $E[U(W_1)]$
• Minimum Mean Extended-Gini (MEG) Coefficient Hedge Ratio	Minimize $\Gamma_v(R_h v)$
• Optimum Mean-MEG Hedge Ratio	Maximize $E[R_h] - \Gamma_v(R_h v)$
• Minimum Generalized Semivariance (GSV) Hedge Ratio	Minimize $V_{\delta,\alpha}(R_h)$

• Maximum Mean-GSV Hedge Ratio	Maximize $E[R_h] - V_{\delta,\alpha}(R_h)$
• Minimum VaR Hedge Ratio over a given time period τ	Minimize $Z_\alpha \sigma_h \sqrt{\tau} - E[R_h]\tau$

Notes:
1. R_h = return on the hedged portfolio,
   $E(R_h)$ = expected return on the hedged portfolio,
   $Var(R_h)$ = variance of return on the hedged portfolio,
   σ_h = standard deviation of return on the hedged portfolio
   Z_α = negative of left percentile at α for the standard normal distribution
   A = risk aversion parameter,
   R_F = return on the risk-free security,
   $E(U(W_1))$ = expected utility of end-of-period wealth,
   $\Gamma_v(R_h v)$ = mean extended-Gini coefficient of R_h,
   $V_{\delta,\alpha}(R_h)$ = generalized semivariance of R_h.
2. With W_1 given by equation (5.17), the maximum expected utility hedge ratio includes the hedge ratio considered by Lence (1995 and 1996).

5.3 Minimum-Variance Hedge Ratio

The most widely-used static hedge ratio is the minimum-variance (MV) hedge ratio. Johnson (1960) derives this hedge ratio by minimizing the portfolio risk, where the risk is given by the variance of changes in the value of the hedged portfolio as follows:

$$Var(\Delta V_H) = C_s^2 Var(\Delta S) + C_f^2 Var(\Delta F) - 2C_s C_f Cov(\Delta S, \Delta F).$$

The MV hedge ratio, in this case, is given by:

$$H_J^* = \frac{C_f}{C_s} = \frac{Cov(\Delta S, \Delta F)}{Var(\Delta F)}. \qquad (5.2a)$$

Alternatively, if we use definition (1a) and use $Var(R_h)$ to represent the portfolio risk, then the MV hedge ratio is obtained by minimizing $Var(R_h)$ which is given by:

$$Var(R_h) = Var(R_s) + h^2 Var(R_f) - 2hCov(R_s, R_f).$$

In this case, the MV hedge ratio is given by:

$$h_J^* = \frac{Cov(R_s, R_f)}{Var(R_f)} = \rho \frac{\sigma_s}{\sigma_f}, \qquad (5.2b)$$

where ρ is the correlation coefficient between R_s and R_f, and σ_s and σ_f are standard deviations of R_s and R_f, respectively.

The attractive features of the MV hedge ratio are that it is easy to understand and simple to compute. However, in general the MV hedge ratio is not consistent with the mean-variance framework since it ignores the expected return on the hedged portfolio. For the MV hedge ratio to be consistent with the mean-variance framework, either the investors need to be infinitely risk-averse or the expected return on the futures contract needs to be zero.

5.4 Optimum Mean-Variance Hedge Ratio

Various studies have incorporated both risk and return in the derivation of the hedge ratio. For example, Hsin *et al.* (1994) derive the optimal hedge ratio that maximizes the following utility function:

$$\underset{C_f}{Max}\, V(E(R_h), \sigma; A) = E(R_h) - 0.5A\sigma_h^2, \qquad (5.3)$$

where A represents the risk aversion parameter. It is clear that this utility function incorporates both risk and return. Therefore, the hedge ratio based on this utility function would be consistent with the mean-variance framework. The optimal number of futures contract and the optimal hedge ratio are respectively given by:

$$h_2 = -\frac{C_f^* F}{C_s S} = -\left[\frac{E(R_f)}{A\sigma_f^2} - \rho \frac{\sigma_s}{\sigma_f}\right]. \qquad (5.4)$$

One problem associated with this type of hedge ratio is that in order to derive the optimum hedge ratio, we need to know the individual's risk aversion parameter. Furthermore, different individuals will choose different optimal hedge ratios, depending on the values of their risk aversion parameter.

Since the MV hedge ratio is easy to understand and simple to compute, it will be interesting and useful to know under what condition the above hedge ratio would be the same as the MV hedge ratio. It can be seen from equations (5.2b) and (5.4) that if $A \to \infty$ or $E(R_f) = 0$, then h_2 would be equal to the MV hedge ratio h_J^*. The first condition is simply a restatement of the infinitely risk-averse individuals. However, the second condition does not impose any condition on the risk-averseness, and this is important. It implies that even if the individuals are not infinitely risk averse, then the MV hedge ratio would be the same as the optimal mean-variance hedge ratio if the expected return on the futures contract is zero (i.e. futures prices follow a simple martingale process). Therefore, if futures prices follow a simple martingale process, then we do not need to know the risk aversion parameter of the investor to find the optimal hedge ratio.

5.5 Sharpe Hedge Ratio

Another way of incorporating the portfolio return in the hedging strategy is to use the risk-return tradeoff (Sharpe measure) criteria. Howard and D'Antonio (1984) consider the optimal level of futures contracts by maximizing the ratio of the portfolio's excess return to its volatility:

$$\underset{C_f}{Max}\, \theta = \frac{E(R_h) - R_F}{\sigma_h}, \tag{5.5}$$

where $\sigma_h^2 = Var(R_h)$ and R_F represents the risk-free interest rate. In this case the optimal number of futures positions, C_f^*, is given by:

$$C_f^* = -C_s \frac{\left(\frac{S}{F}\right)\left(\frac{\sigma_s}{\sigma_f}\right)\left[\frac{\sigma_s}{\sigma_f}\left(\frac{E(R_f)}{E(R_s)-R_F}\right)-\rho\right]}{\left[1-\frac{\sigma_s}{\sigma_f}\left(\frac{E(R_f)\rho}{E(R_s)-R_F}\right)\right]}. \tag{5.6}$$

From the optimal futures position, we can obtain the following optimal hedge ratio:

$$h_3 = -\frac{\left(\frac{\sigma_s}{\sigma_f}\right)\left[\frac{\sigma_s}{\sigma_f}\left(\frac{E(R_f)}{E(R_s)-R_F}\right)-\rho\right]}{\left[1-\frac{\sigma_s}{\sigma_f}\left(\frac{E(R_f)\rho}{E(R_s)-R_F}\right)\right]}. \tag{5.7}$$

Again, if $E\left(R_f\right)=0$, then h_3 reduces to:

$$h_3 = \left(\frac{\sigma_s}{\sigma_f}\right)\rho, \tag{5.8}$$

which is the same as the MV hedge ratio h_J^*.

As pointed out by Chen *et al.* (2001), the Sharpe ratio is a highly non-linear function of the hedge ratio. Therefore, it is possible that equation (5.7), which is derived by equating the first derivative to zero, may lead to the hedge ratio that would minimize, instead of maximizing, the Sharpe ratio. This would be true if the second derivative of the Sharpe ratio with respect to the hedge ratio is positive instead of negative. Furthermore, it is possible that the optimal hedge ratio may be undefined as in the case encountered by Chen *et al.* (2001), where the Sharpe ratio monotonically increases with the hedge ratio.

5.6 Estimation of the Minimum-Variance (MV) Hedge Ratio

The conventional approach to estimating the MV hedge ratio involves the regression of the changes in spot prices on the changes in futures price using the OLS technique (e.g., see Junkus and Lee, 1985). Specifically, the regression equation can be written as:

$$\Delta S_t = a_0 + a_1 \Delta F_t + e_t \,, \tag{5.9}$$

where the estimate of the MV hedge ratio, H_j, is given by a_1. The OLS technique is quite robust and simple to use. However, for the OLS technique to be valid and efficient, assumptions associated with the OLS regression must be satisfied. One case where the assumptions are not completely satisfied is that the error term in the regression is heteroscedastic. This situation will be discussed later.

Another problem with the OLS method, as pointed out by Myers and Thompson (1989), is the fact that it uses unconditional sample moments instead of conditional sample moments, which use currently available information. They suggest the use of the conditional covariance and conditional variance in equation (5.2a). In this case, the conditional version of the optimal hedge ratio (equation (5.2a)) will take the following form:

$$H_J^* = \frac{C_f}{C_s} = \frac{Cov(\Delta S, \Delta F) \,|\, \Omega_{t-1}}{Var(\Delta F) \,|\, \Omega_{t-1}} \,. \tag{5.2a*}$$

Suppose that the current information (Ω_{t-1}) includes a vector of variables (X_{t-1}) and the spot and futures price changes are generated by the following equilibrium model:

$$\Delta S_t = X_{t-1}\alpha + u_t \,,$$

$$\Delta F_t = X_{t-1}\beta + v_t \,.$$

In this case the maximum likelihood estimator of the MV hedge ratio is given by (see Myers and Thompson (1989)):

$$\hat{h} \,|\, X_{t-1} = \frac{\hat{\sigma}_{uv}}{\hat{\sigma}_v^2} \,, \tag{5.10}$$

where $\hat{\sigma}_{uv}$ is the sample covariance between the residuals u_t and v_t, and $\hat{\sigma}_v^2$ is the sample variance of the residual v_t. In general, the OLS estimator obtained from equation (5.9) would be different from the one given by equation (5.10). For the two estimators to be the same, the spot and futures prices must be generated by the following model:

$$\Delta S_t = \alpha_0 + u_t, \quad \Delta F_t = \beta_0 + v_t.$$

In other words, if the spot and futures prices follow a random walk, then with or without drift, the two estimators will be the same.

5.7 Numeric Examples

If $\sigma_s = 0.01$, $\sigma_f = 0.02$, $E(R_f) = 0.02$, $E(R_s) = 0.015$, $A = 200$, $R_F = 0.02$, and $\rho = 0.6$. Then we can use equation (5.2b), (5.4), and (5.7) to calculate the Johnson minimum variance hedge ratio, optimal mean-variance hedge ratio, and Sharp hedge ratio.

Johnson MV hedge ratio = $h_J^* = \dfrac{Cov(R_s, R_f)}{Var(R_f)} = \rho \dfrac{\sigma_s}{\sigma_f} = 0.6 \times \frac{0.01}{0.02} = 0.3$

Optimal MV hedge ratio = $h_2 = -\dfrac{C_f^* F}{C_s S} = -\left[\dfrac{E(R_f)}{A\sigma_f^2} - \rho\dfrac{\sigma_s}{\sigma_f}\right] = -\left[\frac{0.02}{200 \times 0.02^2} - 0.6 \times \frac{0.01}{0.02}\right] = 0.05$

Sharp hedge ratio = $h_3 = -\frac{\left(\frac{\sigma_s}{\sigma_f}\right)\left[\frac{\sigma_s}{\sigma_f}\left(\frac{E(R_f)}{E(R_s)-R_F}\right)-\rho\right]}{\left[1-\frac{\sigma_s}{\sigma_f}\left(\frac{E(R_f)\rho}{E(R_s)-R_F}\right)\right]} = -\frac{\left(\frac{0.01}{0.02}\right)\left[\frac{0.01}{0.02}\left(\frac{0.02}{0.015-0.02}\right)-0.6\right]}{\left[1-\frac{0.01}{0.02}\left(\frac{0.02\times 0.3}{0.015-0.02}\right)\right]} =$

$-\frac{0.5(0.5\times(-4)-0.6)}{1-0.5\times(-2.4)} = \frac{1.3}{2.2} = 0.59$

From this example, we found that if A approaches ∞, then optimal minimum variance hedge ratio equals to 0.3. Similarly, if $E(R_f)$ equals to zero, then Sharpe hedge ratio equals to 0.3. Therefore, we can conclude that both optimal MV hedge ratio and Sharpe hedge ratio are generalized cases of Johnson MV hedge ratio.

5.8 Summary

In this Chapter, we have theoretically and empirically discussed three hedge ratios, i.e. Johnson minimum variance hedge ratio, optimal mean-variance hedge ratio, and Sharp hedge ratio. We also show both optimal mean-variance hedge ratio, and Sharp hedge ratio can be reduced to Johnson minimum variance hedge ratio.