Risk Management for Catastrophe Loss

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Abstract

This study investigates the valuation models for three types of catastrophe-linked instruments: catastrophe bonds, catastrophe equity puts, and catastrophe futures and options. First, it looks into the pricing of catastrophe bonds under stochastic interest rates and examines how reinsurers can apply catastrophe bonds to reduce the default risk. Second, it models and values the catastrophe equity puts that give the (re)insurer the right to sell its stocks at a predetermined price if catastrophe losses surpass a trigger level. Third, this study models and prices catastrophe futures and catastrophe options contracts that are based on a catastrophe index.

Key Words: Catastrophe Risk, Catastrophe Bond, Catastrophe Equity Put, Catastrophe Futures Options, Contingent Claim Analysis.

JEL classification: G20, G28, G21

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1 Introduction

Catastrophic events having low frequency of occurrence but generally high loss severity can easily erode the underwriting capacity of property and casualty insurance and reinsurance companies (P&Cs, hereafter). P&Cs traditionally hedge catastrophe risks by buying catastrophe reinsurance contracts. Because of capacity shortage and constraints in the reinsurance markets, the capital markets develop alternative risk transfer instruments to provide (re)insurance companies with vehicles for hedging their catastrophe risk. These instruments can be broadly classified into three categories: insurance-linked debt contracts (e.g. catastrophe bonds), contingent capital financing instruments (e.g. catastrophe equity puts), and catastrophe derivatives (e.g. catastrophe futures and catastrophe options).

The Chicago Board of Trade (CBOT) launched catastrophe (CAT) futures in 1992 and CAT futures call spreads in 1993 with contract values linked to the loss index compiled by the Insurance Services Office. The CBOT switched to CAT options in 1995 to try to spur growth in the CAT derivatives market, but was unable to generate meaningful activity and ultimately abandoned it in 2000. The CAT bonds, however, have been quite successful with 89 transactions completed, representing $15.53 billion in issuance since the first issue in 1997.[1] Since 1996, several CAT equity put deals were negotiated usually with obligations to purchase stock of $100 million each.

This study looks into the valuation models for these CAT-linked instruments and examines how their values are related to catastrophe risk, terms of the contract, and other key elements of these instruments. The rest of this study is organized into four sections. Section II provides a model to value catastrophe bonds under stochastic interest rates. Section III models and values the catastrophe equity puts with credit risk, and section IV models and prices catastrophe futures and catastrophe options contracts that are based on specified

[1] See MMC Security (2007).

catastrophe indices. Section V investigates how reinsurers can apply catastrophe bonds to reduce their default risk. This study's valuation approach employs the contingent claim analysis, and when a closed-form solution cannot be derived numerical estimates will be computed using the Monte Carlo simulation method.

2 Catastrophe bonds

The CAT bond, which is also named as an "Act of God bond", is a liability-hedging instrument for insurance companies. There are debt-forgiveness triggers in CAT bond provisions, whereby the generic design allows for the payment of interest and/or the return of principal forgiveness, and the extent of forgiveness can be total, partial, or scaled to the size of the loss. Moreover, the debt forgiveness can be triggered by the (re)insurer's actual losses or on a composite index of insurers' losses during a specific period. The advantage of a CAT bond hedge for (re)insurers is that the issuer can avoid the credit risk. The CAT bondholders provide the hedge to the (re)insurer by forgiving existing debt. Thus, the value of this hedge is independent of the bondholders' assets and the issuer has no risk of non-delivery on the hedge. However, from the bondholder's perspective, the default risk, the potential moral hazard behavior, and the basis risk of the issuing firm are critical in determining the value of CAT bonds.

2.1 CAT bond valuation models

Litzenberger et al. (1996) considered a one-year bond with an embedded binary CAT option. The repayment of principal is indexed to the (re)insurer's catastrophe loss (denoted as C_T). This security may be decomposed into two components: (1) long a bond with a face value of F and (2) short a binary call on the catastrophe loss and with a strike price K. Under the assumption that the natural logarithm of the catastrophe loss is normally distributed, the one-year CAT bond can be priced as follows:

$$P_{CAT} = e^{-rT} \times \left(F - \Phi\left[-z_K\right] \times PO_T\right), \tag{1}$$

$$z_K = \frac{\log(K) - u}{\sigma}.$$

Here, r is the risk free interest rate; μ and σ are respectively the mean and standard deviation of $\ln(C_T)$; $\Phi(\cdot)$ denotes the cumulative distribution function for a standard normal random variable; PO_T refers to the option's payout at maturity date, T. For a CAT call option spread, when the (re)insurer's catastrophe loss (C_T) is less than K_1, the bondholder receives a repayment of the entire principal; when the loss is between K_1 and K_2 (where $K_2 > K_1$), the fraction of principal lost is $\frac{(C-K_1)}{(K_2-K_1)}$; and for the loss greater than K_2, the entire principal payment is lost.

This security may be divided into two components: (1) long a bond with an above-market coupon (c) and (2) a CAT call option spread consisting of a short position on the CAT call with a strike price of K_1 and a long position on the CAT call with a strike price of K_2. Under the assumption that the (re)insurer's catastrophe loss is lognormally distributed with mean μ and standard deviation σ, the CAT bond can be priced as follows:

$$P_{CAT} = e^{-rT} \times F \times \left(\begin{array}{c} 1 - \Phi\left[-z_{K_1}\right] \times \left[e^{\mu + \frac{1}{2}\sigma^2} \times \frac{\Phi\left[-z_{K_1} + \sigma\right]}{\Phi\left[-z_{K_1}\right]} - K_1 \right] \\ + \Phi\left[-z_{K_2}\right] \times \left[e^{\mu + \frac{1}{2}\sigma^2} \times \frac{\Phi\left[-z_{K_2} + \sigma\right]}{\Phi\left[-z_{K_2}\right]} - K_2 \right] \end{array} \right) \tag{2}$$

$$z_{K_i} = \frac{\log(K_i) - u}{\sigma}. \ i = 1, 2.$$

Litzenberger et al.(1996) provided a bootstrap approach to price these hypothetical CAT bonds and compared them with the prices calculated under the assumption of the lognormality of catastrophe loss distribution. Zajdenweber (1998) followed Litzenberger et al. (1996), but changed the CAT loss distribution to the stable-Levy distribution. Contrary to Litzenberger et al. (1996) and Zajdenweber (1998), there were a series of attempts to relax the interest rate assumption to be stochastic. For instance, Loubergé et al. (1999) numerically estimated the CAT bond price by assuming the interest rate follows a binomial random process and the catastrophe loss a compound Poisson process.

Lee and Yu (2002) extended the literature and priced CAT bonds with a formal term structure model of Cox et al. (1985). Under the setting that the aggregate loss is a compound Poisson process, a sum of jumps, the aggregate catastrophe loss facing the (re)insurer i can be described as follows:

$$C_{i,t} = \sum_{j=1}^{N(t)} X_{i,j}, \tag{3}$$

where the process $\{N(t)\}_{t\geq 0}$ is the loss number process, which is assumed to be driven by a Poisson process with intensity λ. Terms $X_{i,j}$ denote the amount of losses caused by the jth catastrophe during the specific period for the issuing (re)insurance company. Here, $X_{i,j}$, for $j = 1, 2, ..., N(T)$, are assumed to be mutually independent, identical, and lognormally-distributed variables, which are also independent of the loss number process, and their logarithmic means and variances are μ_i and σ_i^2, respectively.

A discount bond whose payoffs (PO_T) at maturity (i.e. time T) can be specified as follows:

$$PO_T = \begin{cases} F & if \quad C_T \leq K \\ rp \times F & if \quad C_T > K, \end{cases} \tag{4}$$

where K is the trigger level set in the CAT bond provisions, $C_{i,T}$ is the aggregate loss at maturity, rp is the portion of principal needed to be paid to bondholders when the forgiveness trigger has been pulled, and F is the face value of the CAT bond. Under the assumption that the term structure of interest rates is independent of the catastrophe risk, the CAT bond can be priced as follows:

$$P_{CAT} = P_{CIR}(0,T) \times [\sum_{j=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} F^j(K) + rp(1 - \sum_{j=0}^{\infty} e^{-\lambda T} \frac{(\lambda T)^j}{j!} F^j(K)], \tag{5}$$

where

$$F^j(K) = Pr(X_{i,1} + X_{i,2} + ... + X_{i,j} \leq K)$$

denotes the jth convolution of F, and

$$P_{CIR}(0,T) = A(0,T)e^{-B(0,T)r(0)},$$

where

$$A_{CIR}(0,T) = [\frac{2\gamma e^{(\kappa+\gamma)\frac{T}{2}}}{(\kappa+\gamma)(e^{\gamma T}-1)+2\gamma}]^{\frac{2\kappa m}{v^2}}$$
$$B(0,T)_{CIR} = \frac{2(e^{\gamma T}-1)}{(\gamma+\kappa)(e^{\gamma T}-1)+2\gamma}$$
$$\gamma = \sqrt{\kappa^2+2v^2}.$$

Here, κ is the mean-reverting force measurement, and v is the volatility parameter for the interest rate.

2.1.1 Approximating An Analytical Solution

Under the assumption that the catastrophe loss amount is independent and identically lognormally-distributed, the exact distribution of the aggregate loss at maturity, denoted as $f(C_{i,T})$, cannot be known. Lee and Yu (2002) approximated the exact distribution by a lognormal distribution, denoted as $g(C_{i,T})$, with specified moments.[2] Following the approach, the first two moments of $g(C_{i,T})$ are set to be equal to those of $f(C_{i,T})$, which can be written as:

$$\mu_g = E[C_{i,T}] = \lambda T e^{\mu_X + \frac{1}{2}\sigma_X^2} \tag{6}$$

$$\sigma_g^2 = Var[C_{i,T}] = \lambda T e^{2\mu_X + 2\sigma_X^2}, \tag{7}$$

where μ_g and σ_g^2 denote the mean and variance of the approximating distribution $g(C_{i,T})$, respectively. The price of the approximating analytical CAT bond can be shown to be the following:

$$P_{CIR}(0,T)\left[\int_0^K \frac{1}{\sqrt{2\pi}\sigma_g C_{i,T}} e^{-\frac{1}{2}\left(\ln C_{i,T}-\mu_g\right)^2} dC_{i,T} + rp\int_K^\infty \frac{1}{\sqrt{2\pi}\sigma_g C_{i,T}} e^{-\frac{1}{2}\left(\ln C_{i,T}-\mu_g\right)^2} dC_{i,T}\right]. \tag{8}$$

[2] Jarrow and Rudd (1982), Turnbull and Wakeman (1991), and Nielson and Sandmann (1996) used the same assumption in approximating the values of Asian options and basket options.

We report the results of Lee and Yu (2002) in Table 1 to illustrate the difference between the analytical estimates and numerical estimates. Table 1 shows that the values of the approximating solution and the values from the numerical method are very close and within the range of 10 basis points for most cases. In addition, the approximate CAT bond prices are higher than those estimated by the Monte Carlo simulations for a high value of σ_i. This is because the approximate lognormal distribution underestimates the tail probability of losses and this underestimation is more significant when σ_i is high. We also note that the CAT bond price increases with trigger levels and this increment rises with occurrence intensity and loss variance.

2.1.2 Default-Risky CAT Bonds

In order to look into the practical considerations of default risk, basis risk, and moral hazard relating to CAT bonds, Lee and Yu (2002) developed a structural model in which the insurer's total asset value consists of two risk components - interest rate and credit risk. The term credit risk refers to all risks that are orthogonal to the interest rate risk. Specifically, the value of an insurer's assets is governed by the following process:

$$\frac{dV_t}{V_t} = \mu_V dt + \phi dr_t + \sigma_V dW_{V,t}, \tag{9}$$

where V_t is the value of the insurer's total assets at time t; r_t is the instantaneous interest rate at time t; $W_{V,t}$ is the Wiener process that denotes the credit risk; μ_A is the instantaneous drift due to the credit risk; σ_V is the volatility of the credit risk; and ϕ is the instantaneous interest rate elasticity of the insurer's assets.

In the case where the CAT bondholders have priority for salvage over the other debtholders, the default-risky payoffs of CAT bonds can be written as follows:

$$PO_{i,T} = \begin{cases} a * L & \text{if } C_{i,T} \leq K \text{ and } C_{i,T} \leq V_{i,T} - a * L \\ rp * a * L & \text{if } K < C_{i,T} \leq V_{i,T} - rp * a * L \\ Max\left\{V_{i,T} - C_{i,T}, 0\right\} & \text{otherwise,} \end{cases} \tag{10}$$

where $PO_{i,T}$ are the payoffs at maturity for the CAT bond forgiven on the issuing firm's own actual losses; $V_{i,T}$ is the issuing firm's asset value at maturity; $C_{i,T}$ is the issuing firm's aggregate loss at maturity; a is the ratio of the CAT bond's face amount to total outstanding debts (L). According to the payoff structures in $PO_{i,T}$ and the specified asset and interest rate dynamics, the CAT bonds can be valued as follows:

$$P_i = \frac{1}{a * L} E_0^*[e^{-\bar{r}T} PO_{i,T}], \tag{11}$$

where P_i is the default-risky CAT bond price with no basis risk. Term E_0^* denotes expectations taken on the issuing date under risk-neutral pricing measure; $\bar{r}$ is the average risk-free interest rate between issuing date and maturity date; and $\frac{1}{a*L}$ is used to normalize the CAT bond prices for a \$1 face amount.

2.1.3 Moral hazard and basis risk

Moral hazard results from less loss-control efforts by the insurer issuing CAT bonds, since these efforts may increase the amount of debt that must be repaid at the expense of the bondholders' coupon (or principal) reduction. Bantwal and Kunreuther (2000) noted the tendency for insurers to write additional policies in the catastrophe-prone area, spending less time and money in their auditing of losses after a disaster.

Another important element that needs to be considered in pricing a CAT bond is the basis risk. The CAT bond's basis risk refers to the gap between the insurer's actual loss and the composite index of losses that makes the insurer not receive complete risk hedging. The basis risk may cause insurers to default on their debt in the case of high individual loss, but a low index of loss. There is a trade-off between basis risk and moral hazard. If one uses an insurer's actual loss to define the CAT bond payments, then the insurer's moral hazard is reduced or eliminated, but basis risk is created.

In order to incorporate the basis risk into the CAT bond valuation, aggregate catastrophe losses for a composite index of catastrophe losses (denoted as $C_{index,t}$) can be specified as follows:

$$C_{index,t} = \sum_{j=1}^{N(t)} X_{index,j}, \tag{12}$$

where the process $\{N(t)\}_{t\geq 0}$ is the loss number process, which is assumed to be driven by a Poisson process with intensity λ. Terms $X_{index,j}$ denote the amount of losses caused by the jth catastrophe during the specific period for the issuing insurance company and the composite index of losses, respectively. Terms $X_{index,j}$, for $j = 1, 2, ..., N(T)$, are assumed to be mutually independent, identical, and lognormally-distributed variables, which are also independent of the loss number process, and their logarithmic means and variances are μ_{index} and σ^2_{index}, respectively. In addition, the correlation coefficients of the logarithms of $X_{i,j}$ and $X_{index,j}$, for $j = 1, 2, ..., N(T)$ are equal to ρ_X.

In the case of the CAT bond being forgiven on the composite index of losses, the default-risky payoffs can be written as:

$$PO_{index,T} = \begin{cases} a * L & \text{if } C_{index,T} \leq K \text{ and } C_{i,T} \leq V_{i,T} - a * L \\ rp * a * L & \text{if } C_{index,T} > K \text{ and } C_{i,T} \leq V_{i,T} - rp * a * L \\ Max\,\{V_{i,T} - C_{i,T}, 0\} & \text{otherwise,} \end{cases} \tag{13}$$

where $C_{index,T}$ is the value of the composite index at maturity, and a, L, rp, $V_{i,T}$, $C_{i,T}$, and K are the same as defined in equation (10). In the case where the basis risk is taken into account the CAT bonds can be valued as follows:

$$P_{index} = \frac{1}{a * L} E_0^*[\mathrm{e}^{-\bar{r}T} PO_{index,T}], \tag{14}$$

where P_{index} is the default-risky CAT bond price with basis risk at issuing time. Terms E_0^*, $\bar{r}$, and $\frac{1}{a*L}$ are the same as defined in equation (11).

The issuing firm might relax its settlement policy once the accumulated losses fall into the range close to the trigger. This would then cause an increase in expected losses for the next catastrophe. This change in the loss process can be described as follows:

$$\mu_i^{'} = \begin{cases} (1+\alpha)\mu_i & \text{if } (1-\beta)K \leq C_{i,j} \leq K, \\ \mu_i & \text{otherwise,} \end{cases} \tag{15}$$

where $\mu_i^{'}$ is the logarithmic mean of the losses incurred by the $(j+1)th$ catastrophe when the accumulated loss $C_{i,j}$ falls in the specified range, $(1-\beta)K \leq C_{i,j} \leq K$. Term α is a positive constant, reflecting the percentage increase in the mean, and β is a positive constant, which specifies the range of moral hazard behavior.

We expect that both moral hazard and basis risk will drive down the prices of CAT bonds. The results of the effects of moral hazard and basis risk on CAT bonds can be found in Lee and Yu (2002). The significant price differences indicate that the moral hazard is an important factor and should be taken into account when pricing the CAT bonds.[3] A low loss correlation between the firm's loss and the industry loss index subjects the firm to a substantial discount in its CAT bond prices.

3 Catastrophe equity puts

If a insurer suffers a loss of capital due to a catastrophe, then its stock price is likely to fall, lowering the amount it would receive for newly issued stock. Catastrophe equity puts (CatEPut) give insurers the right to sell a certain amount of its stock to investors at a predetermined price if catastrophe losses surpass a specified trigger.[4] Thus, catastrophe equity puts can provide insurers with additional equity capital when they need funds to cover catastrophe losses. A major advantage of catastrophe equity puts is that they make equity funds available at a predetermined price when the insurer needs them the most. However, the insurer that uses catastrophe equity puts faces a credit risk - the risk that the seller of the catastrophe equity puts will not have enough cash available to purchase the insurer's stock at the predetermined price. For the investors of catastrophe equity puts they also face the risk of owning shares of a insurer that is no longer viable.

[3] Bantwal and Kunreuther (2000) also pointed out that moral hazard may explain the CAT bond premium puzzle .

[4] Catastrophe equity puts, or CatEPuts, are underwritten by Centre Re and developed by Aon with Centre Re.

3.1 Catastrophe equity put valuation models

The CatEPut gives the owner the right to issue shares at a fixed price, but that right is only exercisable if the accumulated catastrophe losses exceed a trigger level during the lifetime of the option. Such a contract is a special "double trigger" put option. Cox et al. (2004) valued a CatEPut by assuming that the price of the insurer's equity is driven by a geometric Brownian motion with additional downward jumps of a specified size in the event of a catastrophe.

The price of the insurer's equity can be described as:

$$S_t = S_0 \exp\left(-AN_t + \sigma W_t + \left[\mu_S - \frac{1}{2}\sigma_S^2\right] t\right), \tag{16}$$

where S_t denotes the equity price at time t; $\{W\}_{t\geq 0}$ is a standard Brownian motion; $\{N(t)\}_{t\geq 0}$ is the loss number process, which is assumed to be driven by a Poisson process with intensity λ_S; $A \geq 0$ is the factor to measure the impact of catastrophe on the market price of the insurer's equity; and μ_S and σ_S are respectively the mean and standard deviation of return on the insurer's equity given that no catastrophe occurs during an interval. The option is exercisable only if the number of catastrophes occurring during the lifetime of the contract is larger than a specified number (denoted as n). The payoffs of the CatEPut at maturity can be written as:

$$PO^{CFP} = \begin{cases} K - S_T & if\ S_T < K \text{ and } N_T \geq n \\ 0 & otherwise \end{cases}, \tag{17}$$

where K is the exercise price. This CAT put option can be priced as follows:

$$P^{CFP} = \sum_{j=n}^{\infty} e^{-\lambda_S T} \frac{(\lambda_S T)^j}{j!} \left(Ke^{-rT}\Phi(d_j) - S_0 e^{-Aj+kT}\Phi\left(d_j - \sigma_S\sqrt{T}\right)\right), \tag{18}$$

where

$$k = \lambda_S \left(1 - e^{-A}\right)$$

$$d_j = \frac{\log\left(\frac{K}{S_0}\right) - rT + Aj - kT + \frac{\sigma_S^2 T}{2}}{\sigma_S\sqrt{T}}.$$

Improving upon the assumption of Cox et al. (2004) that the size of the catastrophe is irrelevant, Jaimungal and Wang (2006) assumed that the drop in the insurer's share price depends on the level of the catastrophe losses and valued the CatEPut under a stochastic interest rate. Jaimungal and Wang (2006) modeled the process of the insurer's share price as follows:

$$S_t = S_0 \exp\left(-\alpha \left(L\left(t\right) - \varphi t\right) + X\left(t\right)\right), \tag{19}$$

whereby

$$L\left(t\right) = \sum_{j=1}^{N(t)} l_j,$$

$$dX(t) = \left(\mu_S - \frac{1}{2}\sigma_S^2\right) dt + \sigma_S dW^S\left(t\right),$$

$$dr(t) = \kappa\left(\theta - r\left(t\right)\right) + \sigma_r dW^r\left(t\right),$$

$$d\left[W^S, W^r\right]\left(t\right) = \rho_{S,r} dt,$$

where $W^S\left(t\right)$ and $W^r\left(t\right)$ are correlated Wiener processes driving the returns of the insurer's equity and the short rate, respectively; $L\left(t\right)$ denotes the accumulated catastrophe losses facing the insurer at time t; l_j, for $j = 1, 2, ...$, are assumed to be mutually independent, identical, and distributed variables representing the size of the jth loss with p.d.f $f_L\left(y\right)$ and mean l; $\{N(t)\}_{t\geq 0}$ is a homogeneous Poisson process with intensity λ. The term φt is used to compensate for the presence of downward jumps in the insurer's share price and is chosen as:

$$\varphi = \frac{\lambda}{\alpha} \int_0^\infty \left(1 - e^{-\alpha y}\right) f_L\left(y\right) dy.$$

The parameter α represents the percentage drop in the share price per unit of a catastrophe loss and is calibrated such that:

$$\alpha E\left(l_j\right) = \delta \Longrightarrow \alpha = \frac{\delta}{l}.$$

Since the right is exercisable only if the accumulated catastrophe losses exceed a critical coverage limit during the lifetime of the option, the payoffs of the CatEPut at maturity can be specified as:

$$PO^{JW} = \begin{cases} K - S_T & if\ S_T < K \text{ and } L(T) > \hat{L} \\ 0 & otherwise \end{cases}, \tag{20}$$

where the parameter $\hat{L}$ represents the trigger level of catastrophe losses above which the issuer is obligated to purchase unit shares. Under these settings, the price of the CatEPut at the initial date can be described as follows:

$$P^{JW} = e^{-\lambda T} \sum_{j=1}^{\infty} \frac{(\lambda T)^j}{j!} \int_{\hat{L}}^{\infty} f_L^{(n)}(y) \left\{ KP(0,T)\,\Phi(-d_-(y)) - S_0 e^{-\alpha(y-\varphi T)} \Phi(-d_+(y)) \right\} dy, \tag{21}$$

where $f_L^{(n)}(y)$ represents the n-fold convolution of the catastrophe loss density function $f(L)$;

$$d_{\pm}(y) = \frac{\ln\left(\frac{S_t}{K} P_{Vasicek}(0,T)\right) - \alpha(y - \varphi T) \pm \frac{1}{2}\tilde{\sigma}_r^2}{\tilde{\sigma}_r(0,T)},$$

$$\tilde{\sigma}_r^2(0,T) = \sigma_S^2 T + \frac{2\kappa\rho_{S,r}\sigma_S\sigma_r + \sigma_r^2}{\kappa^2}\left(T - B_{Vasicek}(0,T)\right) - \frac{\sigma_r^2}{2\kappa} B_{Vasicek}^2(0,T).$$

Here, $P(0,T)$ is a T-maturity zero coupon bond in the Vasicek model:

$$P_{Vasicek}(0,T) = \exp\left\{A_{Vasicek}(0,T) - B_{Vasicek}(0,T)\,r(0)\right\},$$

where

$$A(0,T)_{Vasicek} = \left(\theta - \frac{\sigma_r^2}{2\kappa^2}\right)\left(B_{Vasicek}(0,T) - T\right) - \frac{\sigma_r^2}{4\kappa} B_{Vasicek}^2(0,T),$$

$$B_{Vasicek}(0,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa T}\right).$$

3.1.1 Credit risk and CatEPuts

Both Cox et al. (2004) and Jaimungal and Wang (2006) did not consider the effect of credit risk, the vulnerability of the issuer, on the catastrophe equity puts. Here, we follow Cox et al. (2004) to assume that the option is exercisable only if the number of catastrophes occurring during the lifetime of the contract is larger than a specified number, and we develop a model to incorporate the effects of credit risk on the valuation of CatEPuts. Consider an insurer with m_1 shares outstanding that wants to be protected in the event of catastrophe losses by purchasing m_2 units of CatEPuts from a reinsurer. Each CAT put option allows the insurer the right to sell one share of its stock to the reinsurer at a price of K if the insurer's accumulated catastrophe losses during the life of the option exceed a trigger level $\overline{L}$. The payoffs while incorporating the effect of the reinsurer's vulnerability, PO^{LY}, can be written as:

$$\begin{cases} K - S_T & if\ S_T < K \text{ and } P_{L,T} \geq n \text{ and } V_{\mathrm{Re},T} - D_{\mathrm{Re},T} > m_2(K - S_T) \\ (K - S_T) \times \frac{(K - S_T)m_2}{(K - S_T)m_2 + D_{\mathrm{Re},T}} & if\ S_T < K \text{ and } P_{L,T} \geq n \text{ and } V_{\mathrm{Re},T} - D_{\mathrm{Re},T} \leq m_2(K - S_T) \\ 0 & otherwise \end{cases}, \tag{22}$$

where $P_{L,t}$ is the loss number process, which is assumed to be driven by a Poisson process with intensity λ_P; S_t denotes the insurer's share price and can be shown as:

$$S_{i,t} = \frac{V_{i,t} - L_{i,t}}{m_1}, \tag{23}$$

where $V_{i,t}$ and $L_{i,t}$ represent the values of the insurer's assets and liabilities at time t, respectively.

The value dynamics for the insurer's asset and liability are specified as follows:

$$dV_{i,t} = (r + \mu_{V_i})V_{i,t}dt + \sigma_{V_i}V_{i,t}dW_{V_i,t}, \tag{24}$$

$$dL_{i,t} = \left(r + \mu_{L_i} - \lambda_P e^{\mu_{y_i} + \frac{1}{2}\sigma_{y_i}^2}\right) L_{i,t\_}dt + \sigma_{L_i}L_{i,t\_}dW_{L_i,t} + Y_{P_{L_i},t}L_{i,t\_}dP_{L,t}, \tag{25}$$

where r is the risk-free interest rate; μ_{V_i} is the risk premium associated with the insurer's asset risk; μ_{L_i} denotes the risk premium for small shocks in the insurer's liabilities; $W_{V_i,t}$ is a Weiner process denoting the asset risk; $W_{L_i,t}$ is a Weiner process summarizing all continuous shocks that are not related to the asset risk of the insurer; and $Y_{P_{L_i},t}$ is a sequence of independent and identically-distributed positive random variables describing the percentage change in liabilities in the event of a jump. We assume that $\ln(Y_{P_{L_i},t})$ has a normal distribution with mean μ_{y_i} and standard deviation σ_{y_i}. The term $\lambda_P e^{\mu_{y_i}+\frac{1}{2}\sigma^2_{y_i}}$ offsets the drift arising from the compound Poisson component $Y_{P_{L_i},t}L_{i,t_-}dP_{L,t}$.

The value dynamics for the reinsurer's assets ($V_{\text{Re},T}$) and liabilities ($L_{\text{Re},T}$) are specifically governed by the following processes:

$$dV_{\text{Re},t} = (r + \mu_{V_{\mathsf{Re}}})V_{\text{Re},t}dt + \sigma_{V_{\mathsf{Re}}}V_{\text{Re},t}dW_{V_{\mathsf{Re}},t}, \tag{26}$$

$$dL_{\text{Re},t} = \left(r + \mu_{L_{\mathsf{Re}}} - \lambda_P e^{\mu_{y\mathsf{Re}}+\frac{1}{2}\sigma^2_{y\mathsf{Re}}}\right)L_{\text{Re},t_-}dt + \sigma_{L_{\mathsf{Re}}}L_{\text{Re},t_-}dW_{L_{\mathsf{Re}},t} + Y_{P_{L_{\mathsf{Re}}},t}L_{\text{Re},t_-}dP_{L,t}, \tag{27}$$

where $V_{\text{Re},t}$ and $L_{\text{Re},t}$ represent the values of the reinsurer's assets and liabilities at time t, respectively; r is the risk-free interest rate; $\mu_{V_{\mathsf{Re}}}$ is the risk premium associated with the reinsurer's asset risk; $\mu_{L_{\mathsf{Re}}}$ denotes the risk premium for continuous shocks in the insurer's liabilities; $W_{V_{\mathsf{Re}},t}$ is a Weiner process denoting the asset risk; $W_{L_{\mathsf{Re}},t}$ is a Weiner process summarizing all continuous shocks that are not related to the asset risk of the reinsurer; and $Y_{P_{L_{\mathsf{Re}}},t}$ is a sequence of independent and identically-distributed positive random variables describing the percentage change in the reinsurer's liabilities in the event of a jump. We assume that $\ln(Y_{P_{L_{\mathsf{Re}}},t})$ has a normal distribution with mean $\mu_{y\mathsf{Re}}$ and standard deviation $\sigma_{y\mathsf{Re}}$. In addition, assume that the correlation coefficient of $\ln(Y_{P_i,t})$ and $\ln(Y_{P_{L_{\mathsf{Re}}},t})$ is equal to ρ_Y. The term $\lambda_P e^{\mu_{y\,\mathsf{Re}}+\frac{1}{2}\sigma^2_{y\mathsf{Re}}}$ offsets the drift arising from the compound Poisson component $Y_{P_{L_{\mathsf{Re}}},t}L_{\text{Re},t_-}dP_{L,t}$.

According to the payoff structures, the catastrophe loss number process, and the dynamics for the (re)insurer's assets and liabilities specified above, the CatEPut can be valued as follows:

$$P^{LY} = E^*\left[e^{-rT} \times PO^{LY}\right]. \tag{28}$$

Here, $E^*[\bullet]$ denotes expectations taken on the issuing date under a risk-neutral pricing measure.

The CAT put prices are estimated by the Monte Carlo simulation. Table 2 presents the numerical results. It shows that the possibility of a reinsurer's vulnerability (credit risk) drives the put price down dramatically. We also observe that the higher the correlation coefficient of $\ln(Y_{P_i,t})$ and $\ln(Y_{P_{L_{\mathsf{Re}}},t})$ (i.e. ρ_Y) is, the lower the value of the CatEPut will be. This implies that the reinsurer with efficient diversification in providing reinsurance coverage can increase the value of the CatEPut.

4 Catastrophe derivatives

Catastrophe risk for (re)insurers can be hedged by buying exchange-traded catastrophe derivatives such as catastrophe futures, catastrophe futures options, and catastrophe options. Exchange-traded catastrophe derivatives are standardized contracts based on specified catastrophe loss indices. The loss indices reflect the entire P&C insurance industry. The contracts entitle (re)insurers (the buyers of catastrophe derivatives) a cash payment from the seller if the catastrophes cause the index to rise above the trigger specified in the contract.

4.1 Catastrophe derivatives valuation models

A general formula for the catastrophe futures price can be developed as in Cox and Schwebach (1992) as follows:

$$F_t = \frac{1}{Q}\left(AL_t + E\left[Y_t | J_t\right]\right), \tag{29}$$

where Q is the aggregate premium paid for in the catastrophe insurance portfolio. Here, Y_t denotes the losses of the catastrophe insurance portfolio which are reported after the current time t, but included in the settlement value, AL_t is the current amount of catastrophe losses announced by the exchange, and J_t denotes the information available at time t. Cox and Schwebach further derived the catastrophe futures price by assuming Y_t follows a compound Poisson distribution with a intensity parameter λ_Y. The aggregate losses of a catastrophe insurance portfolio would be the sum of a random variable of individual catastrophe losses which are independent and identically distributed. In other words, $Y_t = X_1 + X_2 + ... + X_N$, where X_1, $X_2, ..., X_N$ are mutually independent individual catastrophe losses. According to these assumptions, the futures price can be described as follows:

$$F_t = \frac{1}{Q} AL_t + (T - t)\,\lambda_Y p_1, \tag{30}$$

where p_1 represents the first moment of the individual catastrophe loss distribution, i.e. $p_1 = E(X_i)$. Assuming that the loss of a catastrophe insurance portfolio at maturity (i.e. Y_T) is lognormally distributed, that is, the logarithm of $\frac{AL_T}{AL_t}$ is normally distributed with mean $\mu(T-t)$ and variance $\sigma^2(T-t)$, the futures price can be described as:

$$F_t = \frac{AL_t}{Q} e^{\left(\mu(T-t) + \frac{\sigma^2(T-t)}{2}\right)}. \tag{31}$$

In the case where AL_T is set to be lognormally distributed, Cox and Schwebach (1992) presented the value of a catastrophe futures call option with exercise price x, denoted as C_{CS}, as follows:

$$C_{CS} = \frac{e^{-r(T-t)}}{Q} \left(AL_t e^{\left(\mu + \frac{\sigma^2}{2}\right)(T-t)} \Phi(y_1) - xQ\Phi(y_2) \right), \tag{32}$$

$$y_1 = \frac{\log\left(\frac{AL_t}{xQ}\right) + \mu(T-t) + \frac{\sigma^2(T-t)}{2}}{\sigma\sqrt{T-t}} \text{ and } y_2 = y_1 - \sigma\sqrt{T-t}.$$

Cummins and Geman (1995) used two different processes to describe the instantaneous claim processes during the event quarter and the run-off quarter. They argued that the reporting claims by policyholders are continuous and take only a positive value, hence specifying the instantaneous claim to be a geometric Brownian motion during the run-off quarter. Moreover, they added a jump process to the process during the event quarter. Consequently, the two instantaneous claim processes during the event quarter $\left(t \in \left[0, \frac{T}{2}\right]\right)$ and run-off quarter $\left(t \in \left[\frac{T}{2}, T\right]\right)$ can be respectively specified as follows:

$$\begin{aligned} dc_t &= c_{t^-}\left(\mu_c dt + \sigma_c dW_{c,t}\right) + J_c dN_{c,t} \quad for\ t \in \left[0, \frac{T}{2}\right], \\ dc_t &= c_t\left(\mu_c^{'} dt + \sigma_c^{'} dW_{c,t}\right) \quad for\ t \in \left[\frac{T}{2}, T\right], \end{aligned}$$

where c_t denotes the instantaneous claim which means that the amount of claims reported during a small length of time dt is equal to $c_t dt$. Terms μ_c and $\mu_c^{'}$ represent the mean of the continuous part of the instantaneous claims during the event quarter and run-off quarter, respectively, while σ_c and $\sigma_c^{'}$ represent the standard deviation of the continuous part of the instantaneous claims during the event quarter and run-off quarter, respectively. Term J_c is a positive constant representing the severity of loss jump due to a catastrophe, $N_{c,t}$ is a Poisson process with intensity λ_c, and $W_{c,t}$ is a standard Brownian motion.

Cummins and Geman (1995) derived a formula to value the futures price at time t as follows:

$$\begin{aligned} F_t &= \int_0^t c_s ds + c_t \left(\frac{\exp^{\alpha\left(\frac{T}{2}-t\right)} - 1}{\alpha} + k \right) + J_c \lambda_c \left(\frac{\exp^{\alpha\left(\frac{T}{2}-t\right)} - \alpha\left(\frac{T}{2}-t\right) - 1}{\alpha^2} \right) \\ &\quad + c_0 \exp^{\alpha^{'}\left(\frac{T}{2}-t\right)} \left(\frac{\exp^{\alpha^{'}\frac{T}{2}} - 1}{\alpha^{'}} \right) + J_c \lambda_c \left(\frac{\exp^{\alpha\left(\frac{T}{2}-t\right)} - 1}{\alpha} \right) \left(\frac{\exp^{\alpha^{'}\frac{T}{2}} - 1}{\alpha^{'}} \right), \end{aligned} \tag{33}$$

where $\alpha = \mu_c - \rho\sigma_c$ and $\alpha^{'} = \mu_c^{'} - \rho\sigma_c^{'}$. Here, ρ represents the equilibrium market price of claim level risk and is assumed to be constant over period $[0, T]$. Cummins and Geman (1995) also considered catastrophe call spreads written on the catastrophe loss ratio. The payoffs of European call spreads at maturity T, denoted as $C_{spread}(S, k_1, k_2)$, can be written as follows:

$$C_{spread}(c, k_1, k_2) = Min\left\{ Max\left[100\frac{\int_0^T c_s ds}{Q_c} - k_1, 0 \right], k_2 - k_1 \right\}, \tag{34}$$

where k_1 and k_2 are the exercise prices of the catastrophe call spread and $k_2 > k_1$, while Q_c is the premiums earned for the event quarter. Since no close-form solution can be obtained, the catastrophe call spreads under alternative combinations of exercise prices can be estimated by Monte Carlo simulation. We report the values of 20/40 call spreads estimated by Cummins and Geman (1995) in Table 3 to present the effects of parameter values on the value of catastrophe call spreads.

Chang, Chang and Yu (1996) used the randomized operational time approach to transfer a compound Poisson process to a more tractable pure diffusion process and led to the parsimonious pricing formula of catastrophe call options as a risk-neutral Poisson sum of Black's call prices in information time. Chang et al. (1996) assumed catastrophe futures price changes follow jump subordinated processes in calendar-time. The parent process is assumed to be a lognormal diffusion directed by a homogenous Poisson process as follows:

$$\frac{dX_t}{X_t} = \mu_{X_t} dt + \sigma_{X_t} dW_{X_t}, \tag{35}$$

where μ_{X_t} and σ_{X_t} are the stochastic calendar-time instantaneous mean and variance, respectively, and:

$$\mu_{X_t} dt = \mu_X dn\left(t\right) \tag{36}$$

$$\sigma^2_{X_t} dt = \sigma^2_X dn\left(t\right), \tag{37}$$

where

$$\begin{array}{ll} dn\left(t\right) = 1 & \textit{if the jump occurs once in } dt \textit{ with probability } j_X dt, \textit{ otherwise} \\ dn\left(t\right) = 0 & \textit{with probability } 1 - j_X dt. \end{array}$$

Since the instantaneous mean and variance of calendar-time futures return, $\mu_{X_t} dt$ and $\sigma^2_{X_t} dt$, are linear to random information arrival, the information-time proportional factors, μ_X and σ^2_X, are constant. Substituting equations (36) and (37) into equation (35), the parent process in information time can be transferred into a lognormal diffusion process:

$$\frac{dX_n}{X_n} = \mu_X dt + \sigma_X dW_{X_n}. \tag{38}$$

According to the model, the value of the information-type European catastrophe call option with strike price k, denoted as $c\left(X, n, k\right)$, can be written as follows:

$$c\left(X, n, k\right) = \sum_{m=0}^{\infty} \Gamma\left(m, j_X\right) B\left(X\Phi\left(d_1\right) - k\Phi\left(d_2\right)\right), \tag{39}$$

$$d_1 = \frac{\ln\left(\frac{X}{k}\right) + \frac{1}{2}\sigma_X^2 m}{\sigma_X\sqrt{m}}, d_2 = d_1 - \sigma_X\sqrt{m},$$

where $\Gamma(m,j) = \frac{e^{-j_X(T-t)}[j_X(T-t)]^m}{m!}$ is the Poisson probability mass function with intensity j_x. Moreover, $T-t$ is the option's calendar-time maturity, r is the riskless interest rate, $B = e^{-r(T-t)}$ is the price of a riskless matching bond with maturity $T-t$, and m denotes the information time maturity index.

Chang et al. (1996) followed Barone-Adsei and Whaley (1987) for an analytical approximation of the American extension of the Black formula to get the value of the information-time American catastrophe futures call option with strike price k, denoted as $C(X,n,k)$, as follows:

$$C(X,n,k) = \sum_{m=0}^{\infty} \Gamma(m, j_X) C_B(X,n,k), \tag{40}$$

where

$$C_B(X,n,k) = \begin{cases} e^{-rm}[X\Phi(d_1) - k\Phi(d_2)] + A\left(\frac{X}{X^*}\right)^q, & \textit{where } X \leq X^* \\ X-k, & \textit{where } X > X^* \end{cases}, \tag{41}$$

$$A = \left(\frac{X^*}{q}\right)(1 - B\Phi(d_1(X^*))),$$

$$d_1(X^*) = \frac{\ln\left(\frac{X^*}{X}\right) + \frac{1}{2}\sigma_X^2 m}{\sigma_X\sqrt{m}},$$

$$q = \frac{1+\sqrt{1+4h}}{2}, \text{and } h = \frac{2r}{\sigma_X^2(1-B)}.$$

Here, $C_B(X,n,k)$ represents the American extension of the Black formula based on MacMillan's (1986) quadratic approximation of the American stock options. Moreover, X^* is the critical futures price above where the American futures option should be exercised immediately and is determined by solving:

$$X^* - k = e^{-rm}[X\Phi(d_1) - k\Phi(d_2)] + A\left(\frac{X}{X^*}\right)^q + \left(\frac{X^*}{q}\right)(1 - B\Phi(d_1(X^*))).$$

Since a diffusion is a limiting case of a jump subordinated process when the jump arrival intensity approaches infinity and the jump size simultaneously approaches zero, the pricing model of Black (1976) is a special case of equations (39) and (40).

Table 4 reports the values of information-time American catastrophe call spreads estimated by Chang et al. (1996). It shows that the Black formula underprices the spread for the 40/60 case. However, for the 20/40 case, the Black formula overprices when the maturity is short. Chang et al. (1996) noted that the Black formula is a limiting case of information-time formula, so that the largest mispricing occurs when the jump intensity j is set at a low value.

5 Reinsurance with CAT-linked securities

P&Cs traditionally diversify and transfer catastrophe risk through reinsurance arrangements. The objective of catastrophe reinsurance is to provide protection for catastrophe losses that exceed a specified trigger level. Dassios and Jang (2003) priced stop-loss catastrophe reinsurance contracts while using the Cox process to model the claim arrival process for catastrophes. However, in the case of catastrophic events, reinsurers might not have sufficient capital to cover the losses. Recent studies of the catastrophe reinsurance market have found that these catastrophe events see limited availability of catastrophic reinsurance coverage in the market. (Froot, 1999 and 2001; Harrington and Niehaus, 2003). P&C reinsurers can strengthen their ability in providing catastrophe coverages by issuing CAT-linked instruments. For example, Lee and Yu (2007) developed a model to value the catastrophe reinsurance while considering the issuance of CAT bonds.

The amount that can be forgiven by CAT bondholders when the trigger level has been pulled, δ, can be specified as follows:

$$\delta(C^*) = F_{CAT} - P_{CAT,T}, \tag{42}$$

where $P_{CAT,T}$ is the payoffs of the CAT bond at maturity and is specified as follows:

$$P_{CAT,T} = \begin{cases} F_{CAT} & if\ C^* \leq K_{CATbond} \\ rp \times F_{CAT} & if\ C^* > K_{CATbond} \end{cases}. \tag{43}$$

Here, F_{CAT} is the face value of CAT bonds, and C^* can be the actual catastrophe loss facing the reinsurer (denoted as $C_{i,T}$, specified by equation (3)) or a composite catastrophe index (denoted as $C_{index,T}$, specified by equation (12)) which depends on the provision set by the CAT bond. When the contingent debt forgiven by the CAT bond depends on the actual losses, there is no basis risk. When the basis risk exists, the payoffs of the reinsurance contract remain the same except that the contingent savings from the CAT bond, $\delta(C^*)$, depending on the catastrophic-loss index, become $\delta(C_{index,T})$. Since the debt forgiven by the CAT bond does not depend on the actual loss, the realized losses and savings may not match and may therefore affect the insolvency of the reinsurer and the value of the reinsurance contract in a way that differs from that without basis risk. Here, $K_{CATbond}$ denotes the trigger level set in the CAT bond provision.

In the case where the reinsurer i issues a CAT bond to hedge the catastrophe risk, at maturity the payoffs of the reinsurance contract written by the reinsurer, denoted by $P_{b,T}$, can be described as follows:

$$P_{b,T} = \begin{cases} M - A & if\ \ C_{\text{Re},T} \geq M\ and\ A_{\text{Re},T} + \delta \geq D_{\text{Re},T} + M - A \\ C_{\text{Re},T} - A & if\ \ A \leq C_{\text{Re},T} < M\ and\ A_{\text{Re},T} + \delta \geq D_{\text{Re},T} + C_{\text{Re},T} - A \\ \frac{(M-A)(A_{\mathsf{Re},T}+\delta)}{L_T+M-A} & if\ C_{\text{Re},T} \geq M\ and\ A_{\text{Re},T} + \delta < D_{\text{Re},T} + M - A \\ \frac{(C_{\mathsf{Re},T}-A)(A_{\mathsf{Re},T}+\delta)}{L_T+C_{\mathsf{Re}\,T}-A} & if\ M > C_{\text{Re},T} \geq A\ and\ A_{\text{Re},T} + \delta < D_{\text{Re},T} + C_{\text{Re},T} - A \\ 0 & otherwise, \end{cases} \tag{44}$$

where $A_{\text{Re},T}$ denotes the reinsurer's asset value at time t, which is assumed to be governed by the following process:

$$\frac{dA_{\text{Re},t}}{A_{\text{Re},t}} = \mu_{A_{\mathsf{Re}}} dt + \phi_{A_{\mathsf{Re}}} dr_t + \sigma_{A_{\mathsf{Re}}} dW_{V_{\mathsf{Re}},t}, \tag{45}$$

where $\mu_{A_{\mathsf{Re}}}$ and $\sigma_{A_{\mathsf{Re}}}$ denote respectively the mean and standard deviation of the reinsure's asset return; $\phi_{A_{\mathsf{Re}}}$ is the instantaneous interest rate elasticity of the reinsurer's assets; $C_{\text{Re},T}$

is the catastrophe loss covered by the reinsurance contract; and M and A are respectively the cap and attachment level arranged in the reinsurance contract. In addition to the liability of providing catastrophe reinsurance coverage, the reinsurer also faces a liability that comes from providing reinsurance coverages for other lines. Since the liability represents the present value of future claims related to the non-catastrophic policies, the value of a reinsurer's liability, denoted as $D_{\mathrm{Re},t}$, can be modeled as follows:

$$dD_{\mathrm{Re},t} = (r_t + \mu_{D_{\mathsf{Re}}})D_{\mathrm{Re},t}dt + \phi_{D_{\mathsf{Re}}}D_{\mathrm{Re},t}dr_t + \sigma_{D_{\mathsf{Re}}}D_{\mathrm{Re},t}dW_{D_{\mathsf{Re}},t}, \tag{46}$$

where $\phi_{D_{\mathsf{Re}}}$ is the instantaneous interest rate elasticity of the reinsurer's liabilities.

The continuous diffusion process reflects the effects of interest rate changes and other day-to-day small shocks. Term $\mu_{D_{\mathsf{Re}}}$ denotes the risk premium for the small shock, and $W_{D_{\mathsf{Re}},t}$ denotes the day-to-day small shocks that pertain to idiosyncratic shocks to the capital market. In order to incorporate the effect of the interest rate risk on the reinsurer's assets, the asset value of the reinsurance company is assumed to be governed by the same process as defined in equation (10).

Under the term structure assumption of Cox et al. (1985) the rate on line (ROL) or the fairly-priced premium rate can be calculated as follows:

$$ROL = \frac{1}{M-A} \times E_0^* \left[e^{-\int_0^T r_s ds} \times P_{b,T} \right], \tag{47}$$

where ROL is the premium rate per dollar covered by the catastrophe reinsurance; and E_0^* denotes the expectations taken on the issuing date under risk-neutral pricing measure. Table 5 reports ROLs with and without basis risk calculated by Lee and Yu (2007). When the coefficient of correlation between the individual reinsurer's catastrophe loss and the composite loss index, ρ_X, equals 1, no basis risk exists. The lower the ρ_c is , the higher the basis risk the reinsurer has. The difference of ROLs for a contract with $\rho_X = 1$ and other alternative values is the basis risk premium. We note that the basis risk drives down the value of the reinsurance contract and the impact magnitude increases with the basis risk, catastrophe intensity, and loss volatility. We also note that the basis risk premium decreases with the trigger level and the reinsurer's capital position, but increases with catastrophe occurrence intensity and loss volatility.

6 Conclusion

This study investigates the valuation models for three types of CAT-linked securities: CAT bonds, CAT equity puts, and exchange-traded CAT futures and options. These three new types of securities are capital market innovations which securitize the reinsurance premiums into tradable securities and share the (re)insurers' catastrophe risk with investors.

The study demonstrates how prices of CAT-linked securities can by valued by using a contingent-claim framework and numerical methods via risk-neutral pricing techniques. It begins with introducing a structural model of the CAT bond that incorporates stochastic interest rates and allows for endogenous default risk and shows how its price can be estimated. The model can also evaluate the effect of moral hazard and basis risk related to the CAT bonds. This study then extends the literature by setting up a model for valuing CAT equity puts in which the issuer of the puts is vulnerable. The results show how the values of CAT equity puts change with the issuer's vulnerability and the correlation between the (re)insurer's individual catastrophe risk and the catastrophe index. Both results indicate that the credit risk and the basis risk are important factors in determining CAT bonds and CAT equity puts. This study also compares several models in valuing CAT futures and options. Though differences exist in alternative models, model prices are within reasonable ranges and similar patterns are observed on price relations with the underlying elements. The hedging effect for a reinsurer issuing CAT bonds is also examined.

As long as the threat that natural disasters pose to the financial viability of the P&C

industry continues to exist, there is a need for further innovations on better management of catastrophe risk. The analytical framework in this study in fact provides a platform for future research on CAT losses with more sophisticated products, contacts, and terms.

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Table 1: Default-free CAT Bond Prices:
Approximating Solution vs. Numerical Estimates
No Moral Hazard and Basis Risk

	Approximating Solutions			Numerical Estimates		
	Triggers (K)					
(λ, σ_{X_I})	100	110	120	100	110	120
(0.5,0.5)	0.95112	0.95117	0.95120	0.95119	0.95119	0.95119
(0.5,1)	0.94981	0.95009	0.95031	0.94977	0.95029	0.95062
(0,5,2)	0.92933	0.93128	0.93293	0.92675	0.92903	0.93103
(1,0.5)	0.95095	0.95106	0.95113	0.95119	0.95119	0.95119
(1,1)	0.94750	0.94829	0.94887	0.94825	0.97877	0.94977
(1,2)	0.90559	0.90933	0.91254	0.90273	0.90682	0.91058
(2,0.5)	0.95038	0.95071	0.95091	0.95110	0.95115	0.95119
(2,1)	0.94015	0.94259	0.94441	0.93916	0.94263	0.94492
(2,2)	0.85939	0.86603	0.87183	0.85065	0.85717	0.86378

Notes: All values are calculated assuming bond term $T = 1$, the market price of interest rate $\lambda_r = -0.01$, the initial spot interest rate $r = 5\%$, the long-run interest rate $m = 5\%$, the force of mean-reverting $\kappa = 0.2$, the volatility of the interest rate $v = 10\%$, and the volatility of the asset return that is caused by the credit risk $\sigma_V = 5\%$. All estimates are computed using 20,000 simulation runs.

Table 2: Catastrophe Put Option Prices
With vs. Without Credit Risk

Panel A: $\frac{V_{\mathsf{Re}}}{V_i} = 1$					
	Without Credit Risk	With Credit Risk			
		ρ_y			
λ_P		0.3	0.5	0.8	1
2	0.12787	0.02241	0.02200	0.02140	0.02071
1	0.09077	0.01940	0.01913	0.01864	0.01843
0.5	0.05064	0.00872	0.00873	0.00861	0.00858
0.33	0.02878	0.00451	0.00448	0.00437	0.00433
0.1	0.00391	0.00048	0.00048	0.00043	0.00044
Panel B: $\frac{V_{\mathsf{Re}}}{V_i} = 5$					
2	0.12787	0.03008	0.02918	0.02833	0.02778
1	0.09077	0.02637	0.02616	0.02576	0.02515
0.5	0.05064	0.01333	0.01319	0.01300	0.01280
0.33	0.02878	0.00693	0.00683	0.00664	0.00670
0.1	0.00391	0.00081	0.00081	0.00084	0.00085
Panel C: $\frac{V_{\mathsf{Re}}}{V_i} = 10$					
2	0.12787	0.03126	0.03029	0.02935	0.02860
1	0.09077	0.02730	0.02698	0.02654	0.02611
0.5	0.05064	0.01403	0.01389	0.01357	0.01347
0.33	0.02878	0.00733	0.00722	0.00707	0.00704
0.1	0.00391	0.00087	0.00086	0.00088	0.00091

Note: All values are calculated assuming option term $T = 2$, the number of catastrophe trigger $n = 2$, risk-free interest rate $r = 5\%$, the mean of logarithmic jump magnitude $\mu_{y_i} = -2.3075651$ ($\mu_{y_{\mathsf{Re}}} = -2.3075651$), the standard deviation of logarithmic jump magnitude $\sigma_{y_i} = 0.5$ ($\sigma_{y_{\mathsf{Re}}} = 0.5$), the (re)insurer's initial capital position $\frac{V_i}{L_i} = 1.2$ ($\frac{V_i}{L_i} = 1.2$), the volatility of (re)insurer's assets $\mu_{V_i} = 10\%$ ($\mu_{V_{\mathsf{Re}}} = 10\%$), and the volatility of (re)insurer's pure liabilities $\sigma_{L_i} = 10\%$ ($\sigma_{L_{\mathsf{Re}}} = 10\%$). The catratrophe intensity λ_P is set at 2, 1, 0.5, 0.33, and 0.1. All estimates are computed using 20,000 simulation runs.

Table 3: 20/40 European Catastophe Call Spreads Prices

	σ_s		
Time to maturity	0.2	0.4	0.6
0	3.234	3.842	4.421
0.05	3.192	3.798	4.376
0.10	3.155	3.760	4.336
0.15	3.122	3.727	4.301
0.20	3.095	3.698	4.270
0.25	3.071	3.674	4.244
0.30	3.052	3.653	4.221
0.35	3.035	3.635	4.202
0.40	3.022	3.620	4.185
0.45	3.010	3.608	4.171
0.50	3.002	3.598	4.160

Note: All values are calculated assuming the contract with an expected loss ratio of 20%, the risk-free rate $r = 5\%$, $\lambda_c = 0.5$, $J_c = 0.8$, and the parameters α, α' and μ to be set at 0.1, 0.1, and 0.15, respectively. Strike prices are also in points. Values are quoted in terms of loss ratio percentage points.

Table 4: The Black (B) and Information-time (IT) American Spread Values

Strike/Strike		Maturity(Years)				
(k_1/k_2)	Model	0.025	0.05	0.1	0.25	0.5
20/40	$B\,(IT_\infty)$	0.756	1.067	1.505	2.231	3.048
	IT_{30}	0.549	0.943	1.503	2.583	3.706
	IT_{15}	0.424	0.774	1.322	2.369	3.348
	IT_2	0.154	0.303	0.580	1.288	2.140
40/60	$B\,(IT_\infty)$	0	0	0	0.030	0.228
	IT_{30}	0	0	0	0.056	0.288
	IT_{15}	0	0	0	0.064	0.281
	IT_2	0.013	0.024	0.053	0.146	0.321

Note: All option values are calculated assuming annual volatility $\sigma_X = 60\%$, annual risk-free interest rate $r = 5\%$, and futures price $X = 20$. IT_{30}, IT_{15}, and IT_2 denote information-time values with annual jump arrival intensities $j_X = 30$, 15, and 2, respectively. B denotes the Black value and is dentical to IT_∞, the information-time value when jump arrival intensity is infinity.
The option is capped at 200.

Table 5: Values of Reinsurance Contracts (ROL)
with CAT Bonds and Basis Risk

$K_{CATbond}$	80			100			120		
ρ_c	0.3	0.5	1	0.3	0.5	1	0.3	0.5	1
$(\lambda, \sigma_c, \sigma_{c_{index}})$	$A_{\mathrm{Re}}/D_{\mathrm{Re}}$=1.1								
(0.5,0.5,0.5)	0.00283	0.00283	0.00283	0.00283	0.00283	0.00283	0.00283	0.00283	0.00283
(0.5,1,1)	0.01696	0.01698	0.01709	0.01694	0.01695	0.01701	0.01694	0.01695	0.01701
(0.5,2,2)	0.05335	0.05354	0.05424	0.05327	0.05343	0.05403	0.05326	0.05340	0.05392
(1,0.5,0.5)	0.01067	0.01067	0.01607	0.01066	0.01066	0.01066	0.01066	0.01066	0.01066
(1,1,1)	0.03989	0.03995	0.04018	0.03987	0.03990	0.04002	0.03978	0.03978	0.03986
(1,2,2)	0.10632	0.10663	0.10798	0.10615	0.10640	0.10749	0.10594	0.10620	0.10712
(2,0.5,0.5)	0.04331	0.04331	0.04337	0.04331	0.04331	0.04333	0.04331	0.04331	0.04331
(2,1,1)	0.09952	0.09964	0.10011	0.09931	0.09938	0.09967	0.09925	0.09930	0.09943
(2,2,2)	0.21774	0.21825	0.22703	0.21719	0.21767	0.21981	0.21687	0.21732	0.21910
	$A_{\mathrm{Re}}/D_{\mathrm{Re}}$=1.3								
(0.5,0.5,0.5)	0.00296	0.00296	0.00296	0.00295	0.00295	0.00295	0.00295	0.00295	0.00295
(0.5,1,1)	0.01904	0.01905	0.01918	0.01902	0.01902	0.01908	0.01902	0.01902	0.01906
(0.5,2,2)	0.06184	0.06203	0.06280	0.06177	0.06193	0.06259	0.06175	0.06189	0.06246
(1,0.5,0.5)	0.01123	0.01123	0.01123	0.01123	0.01123	0.01123	0.01123	0.01123	0.01123
(1,1,1)	0.04479	0.04484	0.04510	0.04492	0.04494	0.04508	0.04473	0.04473	0.04482
(1,2,2)	0.12335	0.12365	0.12512	0.12342	0.12367	0.12488	0.12300	0.12326	0.12428
(2,0.5,0.5)	0.04669	0.04669	0.04670	0.04666	0.04666	0.04667	0.04663	0.04663	0.04663
(2,1,1)	0.11265	0.11277	0.11337	0.11249	0.11257	0.11292	0.11237	0.11242	0.11258
(2,2,2)	0.24566	0.24603	0.24791	0.24531	0.24567	0.24732	0.24506	0.24540	0.24677
	$A_{\mathrm{Re}}/D_{\mathrm{Re}}$=1.5								
(0.5,0.5,0.5)	0.00296	0.00296	0.00296	0.00296	0.00296	0.00295	0.00296	0.00296	0.00296
(0.5,1,1)	0.02016	0.02018	0.02030	0.02017	0.02017	0.01906	0.02020	0.02020	0.02023
(0.5,2,2)	0.06887	0.06905	0.06988	0.06882	0.06897	0.06246	0.06887	0.06899	0.06962
(1,0.5,0.5)	0.01130	0.01130	0.01130	0.01130	0.01130	0.01123	0.01130	0.01130	0.01130
(1,1,1)	0.04767	0.04771	0.04800	0.04764	0.04766	0.04482	0.04763	0.04763	0.04773
(1,2,2)	0.13795	0.13824	0.13983	0.13796	0.13820	0.12428	0.13768	0.13793	0.13907
(2,0.5,0.5)	0.04735	0.04735	0.04735	0.04733	0.04733	0.04663	0.04734	0.04734	0.04734
(2,1,1)	0.12066	0.12076	0.12147	0.12051	0.12057	0.11258	0.12055	0.12058	0.12080
(2,2,2)	0.27106	0.27153	0.27417	0.27066	0.27110	0.24677	0.27038	0.27081	0.27268

Note: This table presents ROLs with CAT bond issuance and the payoffs to CAT bonds are linked to a catastrophe loss index. ROLs are calculated and report alternative sets of trigger values ($K_{CATbond}$), catastrophe intensities (λ), catastrophe loss under volatilities ($\sigma_c, \sigma_{c_{index}}$) and the coefficient of correlation between the reinsurer's catastrophe loss and the composite loss index (ρ_X). A_{Re}/D represents the initial asset-liability structure or capital position of the reinsurers.
All estimates are computed using 20,000 simulation runs.