An Analytic Study of Bond Price with Jump Risk

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Abstract

We study bond prices in jump diffusion model incorporating default barrier scheme. We start with a general framework under which the firm value process is assumed to follow Merton's jump diffusion process except that the jump size distribution is arbitrary. We adopt exponential default barrier as in Black and Cox[5]. Moreover, as in Longstaff and Schwartz[16] and Zhou[24], we assume that the bondholders will be paid at the maturity even though a default may have occurred before that time. However, if default occurs, the payoff at maturity date depends on a general writedown function. Under this general setting, we study the properties of bond prices and derive an infinite series expression for them. In particular, we give a non-zero lower bound and upper bound for the credit spreads when the time to maturity tends to zero. It is interesting to compare our results with that obtained using simulation in Zhou[24], and empirically in Jones et.al.[10] and others. See Hilberink and Rogers[9] for related works.

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1 Introduction

There are two basic approaches to model corporate default risks, the structural approach and the reduced form approach. In the classical structural approach such as [18], [5], [16], [13] and [15], the dynamics of a given firm's asset value is assumed to follow a stochastic differential equation:

$$\mathrm{d}V_t = \mu(V_t, t)\mathrm{d}t + \sigma V_t \mathrm{d}W_t.$$

From the continuity of sample paths of diffusion, one obtains the probability that the firm will default near the issuance date is nearly zero. Another implication is that the credit spreads tend to zero as the maturity tends to zero. However, empirical studies indicate that the credit spread is actually quite large even if time to maturity is near zero. For example, in [10], Jones et.al. found that the observed credit spreads are quite large compared to the classical diffusion model; see also [21] for another empirical evidence against the shape of credit spread of the classical model. In [6], Duffie and Lando interpreted this phenomenon is due to information asymmetry.

On the other hand, in [17], the jump processes were stressed to be crucial in the modeling of risky bonds. However, this model is overly simplified since only pure jump processes were applied and the jump amplitudes follow a binomial distribution. Recently, there are some works such as [24], [9] and [22] in which, in addition to a diffusion part, jumps are introduced into the dynamic of the firm's asset value processes. In [24], Merton's jump diffusion model was considered as the underlying firm value process, and the default scheme of Black and Cox that defaults can occur at any time prior to the maturity date was incorporated. Based on the choice of writedown functions, Zhou did not successfully derive a closed form solution for the bond price but provided a Monte Carlo simulation scheme. With his simulation results, he found that by manipulating the parameters, various shapes of credit spreads, default probabilities, and the other characteristics of defaultable bonds found in empirical studies can be recovered. In [9], Hilberink and Rogers generalized the Leland's model; see [14] and [15]. They assumed that the firm's log asset value process is the sum of a Brownian motion with drift and a downward jump compound Poisson process. Although no closed form solution for the price of the perpetual debt, they found the Fourier transform of it. By inverting the Fourier transform, they also found numerically the result that credit spreads do not tend to zero as maturity goes to zero.

In this paper, we consider Merton's jump diffusion model with Black-Cox's default barrier scheme. Similar to the original model of Merton, we allow two-sided jumps at each exponential time. But the jump distribution is now arbitrary instead of a normal distribution as in [19]. As in [16] and [24], we assume that the bondholders will be paid at the maturity even though a default may have occurred before that time. Moreover, we consider the general recovery rate scheme for the risky bonds. In stead of trying to derive a closed form solution for the bond price, we shall write the zero coupon bond as an integral equation which is expanded at the first jump time. The elements of the integral equation are in explicit forms. (See Propositions 3.1 and 3.2 below.) Using these elements, we give an analysis of the bond price. We have the following findings.

- As the firm's initial value tends larger, the price of the bond approximate that of a risk free bond.

- It is "almost" impossible for a solvent firm to default instantaneously.

- Under mild conditions on jump distribution and writedown function, the credit spread bonds has a strictly positive lower bound as time to maturity tends to zero.

(For details, see Theorem 4.1 and Corollary 4.1.) On the other hand, since there cannot occur infinitely many Poisson events in a finite interval, we can use the recursive method to write the bond price as an infinite series in which the series converges uniformly on compacts of space and time(See Theorem 5.1 below).

The structure of this paper is as follows. In Section 2, we recall some basic results in structural modelling of credit risk. In Section 3, we introduce the model and use integral equation techniques to take a first look at the bond price. Based on results in Section 3, we study in Section 4 properties of bond prices and credit spreads. In Section 5, we derive an infinite series expression for the bond prices.

2 Structural Modelling of Credit Risk

In this section, we give a short review of some approaches in the structural modeling of credit risk. For details, see, e.g. [7] and [12].

2.1 The Merton Model

We want to price bonds issued by a firm. In Merton[18], the firm's value is assumed to follow a geometric Brownian motion:

$$\mathrm{d}V_t = \mu V_t \mathrm{d}t + \sigma V_t \mathrm{d}W_t.$$

Assume that there exists a money-market account with a constant riskless rate r. Then it is well known that the no-arbitrage price C_0 at time 0 of a contingent claim paying $C(V_T)$ at time time T is equal to:

$$C_0 = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{-rT} C(V_T)] \tag{2.1}$$

where $\mathbb{Q}$ is the risk-neutral probability measure under which the discounted firm value follows a martingale. More specifically, V is given as:

$$V_t = V_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma W_t^{\mathbb{Q}}\right\},$$

where $W^{\mathbb{Q}}$ is a standard Brownian motion under $\mathbb{Q}$.

Assume that the Modigliani and Miller Theorem holds so that the value of the firm is exogenous and is not influenced by changing the capital structure. Assume that the firm issues a zero coupon bond with face value D which matures at time T. For simplicity, it is assumed that this is the only liability of the firm. Then from the viewpoint of the equity holders, equity value becomes the value of a European call option on the firm's assets with strike price D and maturity date T, that is, the value of the equity is:

$$S_T = \max(V_T - D, 0).$$

On the other hand, the value of the bond holder at time T is,

$$B_T = \min(D, V_T) = D - \max(D - V_T, 0).$$

This means the ownership of the zero coupon bond is equivalent to a long position in a riskless bond with face value D and time to maturity T and a short position in a European put option on firm's assets with strike price D and maturity date T.

Therefore, based on (2.1), if one applies the Black-Scholes formula, one can obtain an explicit pricing formula for the risky bond in Merton model.

2.2 The Merton Model For Jump Diffusion Return

The jump-diffusion model for option pricing in [19] can be directly applied to the pricing of corporate bond. Under the risk-neutral measure $\mathbb{Q}$, the value of the firm follows the dynamics:

$$\mathrm{d}V_t = V_{t-}\left[r\mathrm{d}t + \sigma \mathrm{d}W_t + \mathrm{d}\left(\sum_{i=1}^{N_t} U_i - \lambda\nu t\right)\right], \qquad \mu \in \mathbb{R}, \sigma \in \mathbb{R}_{++}, \tag{2.2}$$

where

- $W = (W_t)$ is a standard Brownian motion.
- $N = (N_t)$ is a Poisson process with intensity λ.
- $\{U_i\}$ is a sequence of independent jump sizes with a common distribution on $(-1, 0) \cup (0, \infty)$ and has mean ν.
- (W_t), (N_t) and $\{U_i\}$ are independent.

The unique(up to indistinguishability) solution to (2.2) is given by

$$V_t = V_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2 - \lambda\nu\right)t + \sigma W_t\right\}\prod_{j=1}^{N_t}(1 + U_j). \tag{2.3}$$

Then we can plug V_T into (2.1) for time zero price of any contingent claim $C(V_T)$ at time T.

In particular, in the case that $1 + U_j$ is lognormal, there is an infinite sum of Black-Scholes type expressions for the price of the risky zero coupon bond.(See Theorem 9.3.1 in [4].)

2.3 The Black-Cox Model

We consider the basic extension of the Merton model in Section 2.1. due to Black and Cox.(See [5].) The idea is to capture the possibility that defaults may occur at any time before maturity for zero coupon bond. In [5], Black and Cox considered a asset value process which, under the risk neutral measure, follows the dynamic:

$$\mathrm{d}V_t = (r - a)V_t \mathrm{d}t + \sigma V_t \mathrm{d}W_t.$$

Here a is the payout rate of the firm. Moreover, they defined the default boundary as:

$$K_t = K_0 \mathrm{e}^{\kappa t}, \qquad 0 \le t \le T, \tag{2.4}$$

where $\kappa \in \mathbb{R}$.(In [16], Longstaff and Schwartz adopted the constant boundary, that is, $\kappa = 0$.) More precisely, the default time is defined by:

$$\tau(T) \triangleq \inf\{0 \le t \le T; V_t \le K_t\} \tag{2.5}$$

with the usual convention that $\inf \emptyset = +\infty$. Then, in Black-Cox model, the payoff at maturity is

$$B(V_T, T) = \min(V_T, D),$$

if there is no default up to time T. If the boundary is hit before or at the maturity of the bond, bond holders will take over the firm and receive the remaining value of the firm at the default time:

$$B(V_{\tau(T)}, \tau(T)) = K_{\tau(T)}.$$

Note that the payoff at maturity if there is no default can be written as

$$D\mathbf{1}_{[\tau(T)>T]} - (D - V_T)^+ \mathbf{1}_{[\tau(T)>T]}.$$

To get the price of the bond payout at maturity if there is no default, we use results on pricing of barrier options. To get the bond payout on the default barrier, one can use the density of the hitting time of Brownian motion and the joint density of Brownian motion and its maximum process.(For details, see [12].)

2.4 Jumps and Default Barriers

It is natural to consider an extension of Black-Cox model in which the dynamics of the asset-value process contains jumps. Note that in this case, defaults can occur either by diffusion or by jumps. Then it is rare to derive closed form solutions for quantities related to defaultable bonds . Indeed, in [24], Zhou proposed an alternative model and gave a simulation scheme for dealing with this situation. More recently, in [9], Hilberink and Rogers showed how to obtain an expression up to a Fourier transform of

$$\mathbb{E}[\mathrm{e}^{-r\tau(\infty)} V_{\tau(\infty)}^{\theta}],$$

where $r > 0$, $\theta > 0$ and $\tau(\infty) = \inf\{t \ge 0; V_t \le K_t\}$. For the method to work, they required that the jumps are only downwards. Also, inversion is not a trivial problem. In the next two sections, as in [24], we consider jump diffusion with two-sided jumps and more general recovery scheme and derive not only some analytic properties of the bond price but also an infinite series expression for it.

3 Integral Equation Approach for Bond Price

3.1 Basic Setup

We assume the setting in Section 2.2 and that the firm's asset value follows (2.3). (From now on, we will write $\mathbb{P}$ for $\mathbb{Q}$.) Assume that the firm issues a zero coupon bond with one dollar face value which matures at $T > 0$ and the term of the bond specifies the bond is of strict priority. In the covenant, the default of the firm is defined as the first time the value of the firm falls below the boundary defined in (2.4). That is, the default time is given by (2.5).

Furthermore, following [16] and [24], we assume that the payoff at T is determined by a function of the firm value at the time of default if there is default or simply the bond pays the full face value if there is no default. In other words, the payoff of the zero coupon bond is given at time T by:

$$\mathbf{1}_{[\tau(T)>T]} + [1-\psi(\log(V_{\tau(T)}/K_{\tau(T)}))]\mathbf{1}_{[\tau(T)\le T]}. \tag{3.1}$$

Here the *writedown function* $\psi(\log x)$ represents the write-down fraction for the bond that is lost due to bankruptcy costs or the others. [Actually, in [24], he considered ψ of the affine-exponential form:

$$\psi(y) = a - b\mathrm{e}^{y}.$$

In the special case that $a = b = 1$ and $K_0 = 1$, $\kappa = 0$, one sees that the payoff of the zero coupon bond at maturity takes the form:

$$\mathbf{1}_{[\tau(T)>T]} + V_{\tau(T)}\mathbf{1}_{[\tau(T)\le T]}.]$$

In literatures, $1-\psi$ is also referred to as recovery rate. It is worth noting that in empirical studies even for the same class of bond issues, the recovery rate $1-\psi$ differs significantly over different time periods and different firms. See [2], [1] and [8].

Set $X_t = \log(V_t/K_t)$. Then

$$\begin{aligned} X_t =& \log(V_0/K_0) + \left(r - \frac{1}{2}\sigma^2 - \lambda\nu - \kappa\right)t + \sigma W_t - Z_t, \qquad (3.2)\\ =& X_0 + ct + \sigma W_t - Z_t, \qquad t \in \mathbb{R}_+, \qquad (3.3)\end{aligned}$$

where $X_0 = x = \log(V_0/K_0)$, $c = r - \frac{1}{2}\sigma^2 - \lambda\nu - \kappa$, and $Z_t \triangleq \sum_{n=1}^{N_t} -\log(1+U_n)$. It follows from (3.1) that the no arbitrage price of the bond is given by

$$D(V_0,T) = \mathrm{e}^{-rT} - \mathrm{e}^{-rT}\mathbb{E}_x\left[\psi(X_{\tau(T)})1_{\{\tau(T)<\infty\}}\right] = \mathrm{e}^{-rT} - \mathrm{e}^{-rT}\Phi(x,T) \tag{3.4}$$

where

$$\Phi(x,t) \triangleq \mathbb{E}_x\left[\psi(X_{\tau(t)})\mathbf{1}(\tau(t)\le t)\right]. \tag{3.5}$$

Note that $\tau(t) = \inf\{0 \le s \le t; X_s \le 0\}$ and $\mathbb{E}_x$ denotes the expectation conditioning on $X_0 = x$ under the risk-neutral measure $\mathbb{P}$. We also write $\mathbb{E}$ for $\mathbb{E}_0$.

3.2 Integral Equation For Φ

For notational convenience, from now on, we will write

$$X_t = X_0 + X_t^c - Z_t$$

with

$$X_t^c = ct + \sigma W_t. \tag{3.6}$$

For any constant $x \in \mathbb{R}$, write $\hat{x} = \frac{x}{\sigma}$, and J_k denotes the k-th epoch time of the compound Poisson process Z with interarrival time S_k, that is,

$$J_k = \sum_{j=1}^{k} S_j.$$

Furthermore, if $G_1, \cdots, G_n$ are random variables on $\mathbb{R}$, we write $F_{G_1,\cdots,G_n}$ for their joint distribution. We preserve F for the distribution of $-\log(1+U_j)$.

We use an integral-equation technique to take a first look at the function Φ. To begin with, write

$$\{\tau(T) \leq T\} = A \cup B \cup C \cup D \tag{3.7}$$

where

$$\begin{aligned}
A =&[J_1 > T \geq \tau(T)] && \text{(No jump up to maturity and default is caused by diffusion)}\\
B =&[T \geq J_1 > \tau(T)] && \text{(Jump occurs up to maturity and default occurs before } J_1\text{)}\\
C =&[T \geq \tau(T) = J_1] && \text{(Jump occurs up to maturity and default occurs at } J_1\text{)}\\
D =&[T \geq \tau(T) > J_1] && \text{(Jump occurs up to maturity and default occurs after } J_1\text{)}
\end{aligned}$$

Note that $\{A, B, C, D\}$ is a partition of $\{\tau(T) \leq T\}$. With these, we define $G_A(x,T) = \mathbb{E}_x\left[\psi(X_{\tau(T)}); A\right]$, and similarly for G_B, G_C and G_D. Before stating our results, we recall basic facts about the distribution of maximum process of and Laplace transform of Brownian motion with drift. For details and proofs, see [11] and [23].

Theorem 3.1 *Let $\alpha \in \mathbb{R}$, $T > 0$, let $W(t;\alpha) = \alpha t + W(t)$ and $M(T;\alpha) = \max_{0\leq t\leq T} W(t;\alpha)$. Then the joint density of $M(T;\alpha)$ and $W(T;\alpha)$ is given by*

$$f_{M(T;\alpha),W(T;\alpha)}(m,w) = \begin{cases} \frac{2(2m-w)}{T\sqrt{2\pi T}}\mathrm{e}^{\alpha w - \frac{1}{2}\alpha^2 T - \frac{1}{2T}(2m-w)^2}, & w \leq m, m \geq 0,\\ 0, & \textit{otherwise.}\end{cases} \tag{3.8}$$

Therefore the density of $M(T;\alpha)$ is given by

$$f_{M(T;\alpha)}(m) = \begin{cases} \frac{2}{\sqrt{2\pi T}}\mathrm{e}^{-\frac{1}{2T}(m-\alpha T)^2} - 2\alpha e^{2\alpha m}\mathcal{N}\left(\frac{-m-\alpha T}{\sqrt{T}}\right), & m \geq 0\\ 0, & \textit{otherwise,}\end{cases} \tag{3.9}$$

and

$$\mathbb{P}\left[M(T;\alpha) \leq m\right] = \begin{cases} \mathcal{N}\left(\frac{m-\alpha T}{\sqrt{T}}\right) - \mathrm{e}^{2\alpha m}\mathcal{N}\left(\frac{-m-\alpha T}{\sqrt{T}}\right), & m \geq 0,\\ 0, & m < 0.\end{cases} \tag{3.10}$$

Here $\mathcal{N}(\cdot)$ is the cumulative distribution function of standard normal distribution.

Proposition 3.1 *We have the following representations of G_A, G_B and G_C:*

$$\begin{aligned}
G_A(x,T) =&\psi(0)\mathrm{e}^{-\lambda T}\left[\mathcal{N}\left(\frac{-\hat{x}-\hat{c}T}{\sqrt{T}}\right) + \mathrm{e}^{-2\hat{c}\hat{x}}\mathcal{N}\left(\frac{-\hat{x}+\hat{c}T}{\sqrt{T}}\right)\right], && (3.11)\\
G_B(x,T) =&\psi(0)\left[\int_0^T \mathcal{N}\left(\frac{-\hat{x}-\hat{c}t}{\sqrt{t}}\right)\mathrm{d}F_{J_1}(t) + \int_0^T \mathrm{e}^{-2\hat{c}\hat{x}}\mathcal{N}\left(\frac{-\hat{x}+\hat{c}t}{\sqrt{t}}\right)\mathrm{d}F_{J_1}(t)\right] && (3.12)\\
G_C(x,T) =& \int_0^T \mathrm{d}F_{J_1}(t)\int_0^\infty \mathrm{d}F(y)\int_0^y \mathrm{d}w\psi(w-y)H(x,w,t), && (3.13)
\end{aligned}$$

where $g(\mu;\sigma^2) = \frac{1}{\sqrt{2\pi\sigma}}\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}$ and

$$H(x,w,t) = g(x-w+ct;t\sigma^2) - \mathrm{e}^{-2\hat{c}\hat{x}}g(x+w-ct;t\sigma^2). \tag{3.14}$$

Proof Note that $\psi(X_{\tau(T)}) = \psi(0)$ on A and B. By independence of $\{W_t; t \in \mathbb{R}_+\}$ and J_1, we obtain

$$\begin{aligned}
G_A(x,T) =& \mathbb{P}[J_1 > T]\psi(0)\mathbb{P}\left[\min_{s\le T} x + cs + \sigma W_s \le 0\right] \\
=& \mathbb{P}[J_1 > T]\psi(0)\mathbb{P}\left[\max_{s\le T} -x - cs - \sigma W_s \ge 0\right] \\
=& \mathbb{P}[J_1 > T]\psi(0)\mathbb{P}\left[\max_{s\le T} -\hat{c}s + W_s \ge \hat{x}\right]
\end{aligned}$$

where the last equality follows from the symmetry of standard Brownian motion. By (3.10), we have

$$\begin{aligned}
\mathbb{P}\left[\max_{s\le T} -\hat{c}s + W_s \ge \hat{x}\right] =& 1 - \mathbb{P}\left[\max_{s\le T} -\hat{c}s + W_s \le \hat{x}\right] \\
=& 1 - \left[\mathcal{N}\left(\frac{\hat{x}+\hat{c}T}{\sqrt{T}}\right) - \mathrm{e}^{-2\hat{c}\hat{x}}\mathcal{N}\left(\frac{-\hat{x}+\hat{c}T}{\sqrt{T}}\right)\right] \\
=& \mathcal{N}\left(\frac{-\hat{x}-\hat{c}T}{\sqrt{T}}\right) + \mathrm{e}^{-2\hat{c}\hat{x}}\mathcal{N}\left(\frac{-\hat{x}+\hat{c}T}{\sqrt{T}}\right).
\end{aligned} \tag{3.15}$$

This completes the proof of (3.11).

We next turn to the proof of (3.12). Again by the independence of $\{W_t; t \in \mathbb{R}_+\}$ and J_1, and the symmetry of standard Brownian motion, we get

$$\begin{aligned}
G_B(x,T) &= \psi(0)\mathbb{P}\left[\min_{s\le J_1} x + cs + \sigma W_s \le 0, J_1 \le T\right] \\
&= \psi(0)\int_0^T \mathbb{P}\left[\min_{s\le t} x + cs + \sigma W_s \le 0\right] \mathrm{d}F_{J_1}(t) \\
&= \psi(0)\int_0^T \mathbb{P}\left[\max_{s\le t} -\hat{c}s + W_s \ge \hat{x}\right] \mathrm{d}F_{J_1}(t).
\end{aligned}$$

Then replacing T with t for (3.15), we get (3.12).

Finally, from independence of $\{W_t; t \in \mathbb{R}_+\}$, $Y_1 \triangleq -\log(1+U_1)$ and J_1,

$$\begin{aligned}
G_C(x,T) &= \int_0^T \mathrm{d}F_{J_1}(t)\mathbb{E}\left[\psi(x + X_t^c - Y_1)\mathbf{1}\left(\min_{s\le t} x + X_s^c > 0, x + X_t^c - Y_1 < 0\right)\right] \\
&= \int_0^T \mathrm{d}F_{J_1}(t)\int_0^\infty \mathrm{d}F(y)\mathbb{E}\left[\psi(x + X_t^c - y)\mathbf{1}\left(\min_{0\le s\le t} x + X_s^c > 0, x + X_t^c - y < 0\right)\right].
\end{aligned}$$

where in the last line we use the fact that $\mathbb{P}\left[\min_{s\le t} x + X_s^c > 0, x + X_t^c - y < 0\right] = 0$ for $y < 0$. Also, observe that, using the symmetry of Brownian motion,

$$\begin{aligned}
&\mathbb{E}\left[\psi(x + X_t^c - y)\mathbf{1}\left(\min_{0\le s\le t} x + X_s^c > 0, x + X_t^c - y < 0\right)\right] \\
=&\mathbb{E}\left[\psi(x - \sigma(-\hat{c}t + W_t) - y)\mathbf{1}\left(\max_{s\le t} -\hat{c}s + W_s \le \hat{x}, \hat{x} - (-\hat{c}t + W_t) - \hat{y} \le 0\right)\right].
\end{aligned}$$

Now, applying the formula of the joint distribution of $W(\alpha;t)$ and $M(\alpha;t)$ with $\alpha = -\hat{c}$, we get for all $t, y > 0$,

$$\begin{aligned}
&\mathbb{E}\left[\psi(x - \sigma(-\hat{c}t + W_t) - y)\mathbf{1}\left(\max_{s\le t} -\hat{c}s + W_s \le \hat{x}, \hat{x} - (-\hat{c}t + W_t) - \hat{y} \le 0\right)\right] \\
&= \int \mathrm{d}w \int_{w^+}^{\infty} \mathrm{d}m \psi(x - \sigma w - y)\mathbf{1}(m \le \hat{x}, \hat{x} - w - \hat{y} \le 0)\frac{2(2m - w)}{t\sqrt{2\pi t}}\mathrm{e}^{-\hat{c}w - \frac{1}{2}\hat{c}^2 t - \frac{1}{2t}(2m-w)^2} \\
&= \int_{\hat{x}-\hat{y}}^{\hat{x}} \mathrm{d}w \psi(x - \sigma w - y)\mathrm{e}^{-\hat{c}w - \frac{1}{2}\hat{c}^2 t} \int_{w^+}^{\hat{x}} \mathrm{d}m \frac{2(2m - w)}{t\sqrt{2\pi t}}\mathrm{e}^{-\frac{1}{2t}(2m-w)^2} \\
&= \int_{\hat{x}-\hat{y}}^{\hat{x}} \mathrm{d}w \psi(x - \sigma w - y)\mathrm{e}^{-\hat{c}w - \frac{1}{2}\hat{c}^2 t} \int_{|w|}^{2\hat{x}-w} \mathrm{d}m \frac{m}{t\sqrt{2\pi t}}\mathrm{e}^{-\frac{1}{2t}m^2} \\
&= \int_{\hat{x}-\hat{y}}^{\hat{x}} \mathrm{d}w \psi(x - \sigma w - y)\mathrm{e}^{-\hat{c}w - \frac{1}{2}\hat{c}^2 t} \frac{1}{\sqrt{2\pi t}}\left[\mathrm{e}^{-\frac{w^2}{2t}} - \mathrm{e}^{-\frac{(2\hat{x}-w)^2}{2t}}\right].
\end{aligned}$$

Note that

$$\begin{aligned}
(2\hat{x} - w)^2 + 2t\hat{c}w + \hat{c}^2 t^2 &= 4\hat{x}^2 - 4\hat{x}w + w^2 + 2t\hat{c}w + \hat{c}^2 t^2 \\
&= (4\hat{x}^2 + \hat{c}^2 t^2 + w^2 - 4\hat{x}\hat{c}t + 2\hat{c}tw - 4\hat{x}w) + 4\hat{x}\hat{c}t \\
&= (2\hat{x} - \hat{c}t - w)^2 + 4\hat{x}\hat{c}t.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\hat{x}-\hat{y}}^{\hat{x}} \mathrm{d}w \psi(x - \sigma w - y)\mathrm{e}^{-\hat{c}w - \frac{1}{2}\hat{c}^2 t} \frac{1}{\sqrt{2\pi t}}\left[\mathrm{e}^{-\frac{w^2}{2t}} - \mathrm{e}^{-\frac{(2\hat{x}-w)^2}{2t}}\right] \\
&= \int_{\hat{x}-\hat{y}}^{\hat{x}} \mathrm{d}w \psi(x - \sigma w - y)\frac{1}{\sqrt{2\pi t}}\left[\mathrm{e}^{-\frac{(w+\hat{c}t)^2}{2t}} - \mathrm{e}^{-2\hat{x}\hat{c}} e^{-\frac{(2\hat{x}-\hat{c}t-w)^2}{2t}}\right] \\
&= \int_0^y \mathrm{d}w \psi(w - y)\frac{1}{\sigma\sqrt{2\pi t}}\left[\mathrm{e}^{-\frac{(x-w+ct)^2}{2t\sigma^2}} - \mathrm{e}^{-2\hat{x}\hat{c}} e^{-\frac{(x+w-ct)^2}{2t\sigma^2}}\right] \\
&= \int_0^y \psi(w - y)\left[g(x - w + ct; t\sigma^2) - \mathrm{e}^{-2\hat{x}\hat{c}} g(x + w - ct; t\sigma^2)\right]\mathrm{d}w.
\end{aligned}$$

This gives (3.13). $\Box$

Remark. In case that Y_1 concentrates heavily on (M, ∞) for some large $M > 0$, the firm is not easy to survive up to the first jump time J_1 which occurs prior to maturity date T. Therefore one has $\mathbb{P}_x(D)$ is very small, and it is tempted to use $G_A + G_B + G_C$ to estimate Φ. $\Box$

To calculate G_D, we recall the strong Markov property of Lévy processes. For details, see, e.g., [3] or [20].

Theorem 3.2 *Let $\{\mathcal{F}_t\}$ be the usual augmentation of the natural filtration $\sigma(X_s; 0 \le s \le t)$. Then for any stopping time τ of $\mathcal{F}_t$ and any nonnegative $Z \in \mathcal{F}_\infty$*

$$\mathbb{E}_x[Z \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}_{X_\tau}[Z], \qquad \text{on } \{\tau < \infty\}.$$

Here θ is the shift operator, that is, $X \circ \theta_t(s) = X_{t+s}$ for all $s \ge 0$. Moreover, on $\{\tau < \infty\}$

$$\{X_{\tau+t} - X_\tau; t \in \mathbb{R}_+\} \qquad \text{is independent of } \mathcal{F}_\tau.$$

Proposition 3.2 *We have*

$$G_D(x,T) = \mathbb{E}_x\left[\mathbf{1}_D \Phi(X_{J_1}, T-J_1)\right] = \int_0^T \mathrm{d}F_{J_1}(t) \int \mathrm{d}F(y) \int_{y^+}^\infty \mathrm{d}w \Phi(w-y, T-t) H(x,w,t),$$

where H is defined in (3.14).

Proof Since every jump is observable($\int_{\mathbb{R}\setminus\{0\}} dF(y) = 1$), every exponential time J_n is a stopping time. Note that

$$\widetilde{D} = \left[J_1 \leq T, \min_{s\leq J_1} X_s > 0\right] \in \mathcal{F}_{J_1}.$$

Therefore, by Strong Markov Property, we have

$$G_D(x,T) = \mathbb{E}_x\left[\mathbb{E}_x\left[\mathbf{1}_{\widetilde{D}} \psi(X_{\tau(T)}) \mathbf{1}(\tau(T) \leq T) | \mathcal{F}_{J_1}\right]\right] = \mathbb{E}_x\left[\mathbf{1}_{\widetilde{D}} \Phi(X_{J_1}, T-J_1)\right].$$

Recall that $Y_1 = -\log(1+U_1)$. Therefore, by the independence of J_1, Y_1, and $\{W_t; t \geq 0\}$, we get

$$\begin{aligned} G_D(x,T) &= \mathbb{E}\left[\mathbf{1}\left(J_1 \leq T, \min_{s\leq J_1} x + X_s^c > 0, x + X_{J_1}^c - Y_1 > 0\right) \Phi(x + X_{J_1}^c - Y_1, T - J_1)\right] \\ &= \int_0^T \mathrm{d}F_{J_1}(t) \int \mathrm{d}F(y) \mathbb{E}\left[\mathbf{1}\left(\min_{s\leq t} x + X_s^c > 0, x + X_t^c - y > 0\right) \Phi(x + X_t^c - y, T-t)\right]. \end{aligned}$$

We compute the integrand. Using the symmetry of Brownian motion, we get

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}\left(\min_{s\leq t} x + X_s^c > 0, x + X_t^c - y > 0\right) \Phi(x + X_t^c - y, T-t)\right] \\ =&\mathbb{E}\left[\mathbf{1}\left(\max_{s\leq t} -x - cs + \sigma W_s < 0, -x - ct + \sigma W_t + y < 0\right) \Phi(x + ct - \sigma W_t - y, T-t)\right] \\ =&\mathbb{E}\left[\mathbf{1}\left(\max_{s\leq t} -\hat{x} - \hat{c}s + W_s < 0, -\hat{x} - \hat{c}t + W_t + y < 0\right) \Phi(x - \sigma(-\hat{c}t + W_t) - y, T-t)\right]. \end{aligned}$$

Using the joint distribution of $M(-\hat{c}, t)$ and $-\hat{c}t + W_t$, we get

$$\begin{aligned} &\mathbb{E}\left[\mathbf{1}\left(\max_{s\leq t} -\hat{x} - \hat{c}s + W_s < 0, -\hat{x} - \hat{c}t + W_t + y < 0\right) \Phi(x - \sigma(-\hat{c}t + W_t) - y, T-t)\right] \\ =& \int_{-\infty}^{(\hat{x}-\hat{y})\wedge\hat{x}} \mathrm{d}w \int_{w^+}^{\hat{x}} \mathrm{d}v \Phi(x - \sigma w - y, T-t) \frac{2(2v-w)}{t\sqrt{2\pi t}} \exp\left\{-\hat{c}w - \frac{1}{2}\hat{c}^2 t - \frac{1}{2t}(2v-z)^2\right\} \\ =& \int_{-\infty}^{(\hat{x}-\hat{y})\wedge\hat{x}} \mathrm{d}w \Phi(x - \sigma w - y, T-t) \exp\left\{-\hat{c}w - \frac{1}{2}\hat{c}^2 t\right\} \int_{w^+}^{\hat{x}} \frac{2(2v-w)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2v-z)^2}{2t}\right\} \mathrm{d}v. \end{aligned}$$

Similar to the calculation of G_C, we have

$$\begin{aligned} &\int_{-\infty}^{(\hat{x}-\hat{y})\wedge\hat{x}} \mathrm{d}w \Phi(x - \sigma w - y, T-t) \exp\left\{-\hat{c}w - \frac{1}{2}\hat{c}^2 t\right\} \int_{w^+}^{\hat{x}} \frac{2(2v-w)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2v-z)^2}{2t}\right\} \mathrm{d}v \\ =& \int_{-\infty}^{(\hat{x}-\hat{y})\wedge\hat{x}} \Phi(x - \sigma w - y, T-t) \left[g(w + \hat{c}t; t) - g(w + t\hat{c} - 2\hat{x}; t) \mathrm{e}^{-2\hat{c}\hat{x}}\right] \mathrm{d}w \\ =& \int_{y^+}^\infty \Phi(w - y, T-t) \left[g(w - x - ct; t\sigma^2) - g(w + x - ct; t\sigma^2) \mathrm{e}^{-2\hat{c}\hat{x}}\right] \mathrm{d}w, \end{aligned}$$

where we use the change of variable $x - \sigma w \mapsto w$ in the last equation. This completes the proof of the proposition. □

4 Some Properties of Bond Price

We now use the integral equation approach in Section 3 to study properties of the bond price defined in (3.4). First, to fix idea, we adopt from [12] and [18] the following definition of yield spreads and credit spreads.

Definition 4.1 *For the bond price defined in (3.4), the* promised yield for maturity T *is given by the formula*

$$y(V_0, T) = \frac{1}{T} \log \left(\frac{1}{D(V_0, T)} \right)$$

and the credit spread for maturity T *by:*

$$s(V_0, T) = y(V_0, T) - r.$$

One get immediately from the definition of yield spread that

$$D(V_0, T) \mathrm{e}^{y(V_0, T)T} = 1.$$

Note that 1 is the face value of the bond.

Lemma 4.1 *For all $x > 0$ and $y > 0$, the function*

$$t \mapsto \mathbb{E} \left[\psi(x + X_t^c - y) \mathbf{1} \left(\min_{s \leq t} x + X_s^c > 0, x + X_t^c - y < 0 \right) \right] \tag{4.1}$$

is continuous on $\mathbb{R}_{++}$.

Proof Recall that in the proof of Proposition 3.1, we have

$$\mathbb{E} \left[\psi(x + X_t^c - y) \mathbf{1} \left(\min_{s \leq t} x + X_s^c > 0, x + X_t^c - y < 0 \right) \right] = \int_0^y \psi(w - y) H(x, w, t) \mathrm{d}w,$$

where H is given by (3.14). From this, it follows easily that the function in (4.1) is continuous. This completes the proof. □

Lemma 4.2 *Assume $x > 0$. Then*

$$\lim_{T \to 0+} \frac{\partial}{\partial T} \mathbb{P} \left[\min_{s \leq T} x + cs + \sigma W_s < 0 \right] = 0. \tag{4.2}$$

Also, for all $n \in \mathbb{N}$, we have

$$\mathbb{P} \left[\min_{t \leq T} x + ct + \sigma W_s < 0 \right] = o(T^n), \qquad \text{as } T \to 0+. \tag{4.3}$$

Proof Firstly, we prove (4.2). By the symmetry of Brownian motion, we have

$$\mathbb{P} \left[\min_{s \leq T} x + cs + \sigma W_s \leq 0 \right] = \mathbb{P} \left[\max_{s \leq T} -cs + \sigma W_s \geq x \right] = \mathbb{P} \left[\max_{s \leq T} -\hat{c}s + W_s \geq \hat{x} \right].$$

Note that $x > 0$. Therefore, by (3.10), we have

$$\mathbb{P} \left[\max_{s \leq T} -\hat{c}s + W_s \geq \hat{x} \right] = \mathcal{N} \left(-\frac{\hat{x} + \hat{c}T}{\sqrt{T}} \right) + \mathrm{e}^{-2\hat{c}\hat{x}} \mathcal{N} \left(\frac{-\hat{x} + \hat{c}T}{\sqrt{T}} \right)$$

which converges to 0 as $T \to 0^+$. Recall that $g(x;\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}$. We have

$$
\begin{aligned}
&\frac{\partial}{\partial T}\left[\mathcal{N}\left(-\frac{\hat{x}+\hat{c}T}{\sqrt{T}}\right) + e^{-2\hat{c}\hat{x}}\mathcal{N}\left(\frac{-\hat{x}+\alpha T}{\sqrt{T}}\right)\right] \\
=&-g\left(\frac{-(\hat{x}+\hat{c}T)}{\sqrt{T}};1\right)\frac{\sqrt{T}\hat{c}-(\hat{x}+\hat{c}T)\frac{1}{2}T^{-1/2}}{T} \\
&+\mathrm{e}^{-2\hat{c}\hat{x}}g\left(\frac{-\hat{x}+\hat{c}T}{\sqrt{T}};1\right)\frac{\sqrt{T}\hat{c}-(-\hat{x}+\hat{c}T)\frac{1}{2}T^{-1/2}}{T} \\
=&\frac{1}{2}\left[-g\left(\frac{-(\hat{x}+\hat{c}T)}{\sqrt{T}};1\right)\frac{\hat{c}T-\hat{x}}{T^{3/2}} + g\left(\frac{\hat{x}+\hat{c}T}{\sqrt{T}};1\right)\frac{\hat{c}T+\hat{x}}{T^{3/2}}\right] \\
=&g\left(\frac{\hat{x}+\hat{c}T}{\sqrt{T}};1\right)\frac{\hat{x}}{T^{3/2}} = \frac{1}{\sqrt{2\pi}}\hat{x}\exp\left\{-\frac{1}{2}\left(\frac{\hat{x}}{\sqrt{T}}+\hat{c}\sqrt{T}\right)^2\right\}\frac{1}{T^{3/2}}.
\end{aligned}
$$

Observe that for all $T > 0$

$$
0 \le \frac{1}{\sqrt{2\pi}}\hat{x}\exp\left\{-\frac{1}{2}\left(\frac{\hat{x}}{\sqrt{T}}+\hat{c}\sqrt{T}\right)^2\right\}\frac{1}{T^{3/2}} \le C\exp\left\{-\frac{\hat{x}^2}{2T}\right\}\frac{1}{T^{3/2}},
$$

for some constant $C > 0$ independent of $T > 0$. This implies that (4.2) holds.

We prove (4.3). Let $n \in \mathbb{N}$. Then, by L'Hôspital's rule,

$$
\begin{aligned}
&\lim_{T\to 0+}\frac{\mathbb{P}\left[\min_{s\le T} x + cs + \sigma W_s < 0\right]}{T^n} \\
\le&\lim_{T\to 0+}\frac{\frac{\partial}{\partial T}\mathbb{P}\left[\min_{s\le T} x + cs + \sigma W_s < 0\right]}{nT^{n-1}} \\
\le&\lim_{T\to 0+} C\exp\left\{-\frac{\hat{x}^2}{2T}\right\}\frac{1}{nT^{(n-1)+3/2}} = 0.
\end{aligned}
$$

We have completed the proof.

□

Theorem 4.1 *Let $D(V_0,T)$ be the bond price defined in (3.4) and set $x = \log(V_0/K_0)$. We have the following analytic properties of bond price.*
(a) As the initial asset value becomes infinity, the bond is essentially riskless: for all $T > 0$,

$$
\lim_{V_0\to\infty} D(V_0,T) = \mathrm{e}^{-rT}.
$$

(b) It is "almost" impossible for a solvent firm to default instantaneously: for all $V_0 > 0$,

$$
\lim_{T\to 0+}\mathbb{P}_x[\tau(T) \le T] = 0.
$$

(c) Assume that the writedown function ψ is continuous and $0 \le \psi \le 1$. Moreover, x is a continuity point of F (i.e., $\int_{\{x\}} \mathrm{d}F(y) = 0$.) Then

$$
\lambda \ge \limsup_{T\to 0+} s(V_0,T) \ge \liminf_{T\to 0+} s(V_0,T) \ge \lambda\int_x^\infty \psi(x-y)\mathrm{d}F(y). \tag{4.4}
$$

In particular, if $\psi \equiv 1$ and $\mathbb{P}[Y_1 > x] > 0$, we have a strictly positive credit spread for zero maturity.

Proof We prove (a) first. Since the writedown function ψ is bounded, by (3.4) and (3.5), it suffices to show that

$$\lim_{x\to\infty} \mathbb{P}_x\left[\tau(T) \le T\right] = 0. \tag{4.5}$$

Now since $X = (X_t)$ is cádlág, it is clear that, for fixed T,

$$\tau(T,x) \triangleq \inf\{0 \le t \le T; x + X_t^c - X_t^d \le 0\} \to \infty, \qquad x \uparrow \infty.$$

This shows (4.5).

Next, consider (b). Write

$$\mathbb{P}_x\left[\tau(T) \le T\right] = \mathbb{P}_x(A) + \mathbb{P}_x(B \cup C \cup D). \tag{4.6}$$

where $\{A, B, C, D\}$ is the partition of $\{\tau(T) \le T\}$ in (3.7). For the second term on the right hand side of (4.6), we have

$$P_x(B \cup C \cup D) = P_x\left[\tau(T) \le T, J_1 \le T\right] \le \mathbb{P}[J_1 \le T] = 1 - e^{-\lambda T} \to 0, \qquad T \to 0+. \tag{4.7}$$

On the other hand, we write

$$\mathbb{P}_x(A) = \mathbb{P}_x\left[\tau(T)) \le T, J_1 > T\right] = \mathbb{P}[J_1 > T]\mathbb{P}\left[\min_{s\le T} x + cs + \sigma W_s \le 0\right].$$

By (4.3), we get

$$\mathbb{P}_x(A) \le \lim_{T\to 0+} \mathbb{P}\left[\min_{s\le T} x + cs + \sigma W_s \le 0\right] = 0. \tag{4.8}$$

Combining (4.7) and (4.8), we get (b).

Finally, consider (c). We firstly estimate $\liminf_{T\to 0+} s(V_0, T)$. Now,

$$\begin{aligned}
\liminf_{T\to 0+} s(V_0, T) &= \liminf_{T\to 0+} \left[\frac{1}{T}\log\left(\frac{1}{D(V_0,T)}\right) - r\right] \\
&= \liminf_{T\to 0+} \left[\frac{1}{T}\log\left(\frac{\mathrm{e}^{rT}}{1-\Phi(x,T)}\right) - r\right].
\end{aligned}$$

(Note that by (b), $\Phi(x,T) \to 0$ as $T \to 0+$). On the other hand, since $0 \le \psi \le 1$, we have $\Phi(x,T) \ge G_C(x,T)$. Therefore,

$$\begin{aligned}
\liminf_{T\to 0+} s(V_0, T) &= \liminf_{T\to 0+} \frac{1}{T}\log\left(\frac{1}{1-\Phi(x,T)}\right) \\
&\ge \lim_{T\to 0+} \frac{1}{T}\log\left(\frac{1}{1-G_C(x,T)}\right).
\end{aligned}$$

Since $0 \le G_C(x,T) \le \Phi(x,T) \to 0$ as $T \to 0+$, we have, using L'Hôspital's Rule,

$$\begin{aligned}
&\lim_{T\to 0+} \frac{1}{T}\log\left(\frac{1}{1-G_C(x,T)}\right) \\
&= \lim_{T\to 0+} \frac{\frac{\partial G_C}{\partial T}(x,T)}{1-G_C(x,T)} \\
&= \lim_{T\to 0+} \frac{\partial G_C}{\partial T}(x,T).
\end{aligned}$$

By (3.13), Lemma 4.1 and Fundamental Theorem of Calculus, we obtain

$$\frac{\partial G_C}{\partial T}(x,T) = \lambda e^{-\lambda T} \int_0^\infty \mathrm{d}F(y)\mathbb{E}\left[\psi(x+X_T^c-y)\mathbf{1}\left(\min_{s\le T} x+X_s^c>0, x+X_T^c-y<0\right)\right].$$

Recall that ψ is continuous and $\int_{\{x\}} \mathrm{d}F(y)=0$. Therefore,

$$\begin{aligned}\liminf_{T\to 0+} s(V_0,T) \ge \lim_{T\to 0+}\frac{\partial G_C}{\partial T}(x,T) &= \lambda\int_0^\infty \psi(x-y)\mathbf{1}(x-y<0)\mathrm{d}F(y)\\ &= \lambda\int_x^\infty \psi(x-y)\mathrm{d}F(y).\end{aligned}$$

This proves the lower bound of (4.4).

Next, we show that $\lambda \ge \limsup_{T\to 0+} s(V_0,T)$. Using the partition $\{A,B,C,D\}$ of $\{\tau(T)\le T\}$ as in (3.7), we have

$$\Phi(x,T)=\mathbb{E}_x\left[\psi(X_{\tau(T)}); B\cup C\cup D\right]+G_A(x,T).$$

Since $\mathbb{P}(B\cup C\cup D)\le \mathbb{P}[J_1\le T]=1-\mathrm{e}^{-\lambda T}$ and $\psi\le 1$, we have

$$\mathbb{E}_x\left[\psi(X_{\tau(T)}); B\cup C\cup D\right]\le 1-\mathrm{e}^{-\lambda T}.$$

This implies that

$$\Phi(x,T)\le (1-\mathrm{e}^{-\lambda T})+G_A(x,T).$$

Therefore

$$\begin{aligned}\limsup_{T\to 0+} s(V,T) &= \limsup_{T\to 0+}\frac{1}{T}\log\left(\frac{1}{1-\Phi(x,T)}\right)\\ &\le \lim_{T\to 0+}\frac{1}{T}\log\left(\frac{1}{\mathrm{e}^{-\lambda T}-G_A(x,T)}\right).\end{aligned}$$

Note that $G_A(x,T)\le\Phi(x,T)\to 0$ as $T\to 0+$ and, hence, $\lim_{T\to 0+}\left(\mathrm{e}^{-\lambda T}-G_A(x,T)\right)=1$. By L'Hôspital's Rule, we have

$$\begin{aligned}\lim_{T\to 0+}\frac{1}{T}\log\left(\frac{1}{e^{-\lambda T}-G_A(x,T)}\right) &= \lim_{T\to 0+}\frac{-\left(-\lambda e^{-\lambda T}-\frac{\partial G_A}{\partial T}(x,T)\right)}{e^{-\lambda T}-G_A(x,T)}\\ &= \lim_{T\to 0+}\left(\lambda+\frac{\partial G_A}{\partial T}(x,T)\right).\end{aligned}$$

It remains to compute $\lim_{T\to 0+}\frac{\partial G_A}{\partial T}(x,T)$. We know from the expression of $G_A(x,T)$ that

$$G_A(x,T)=\psi(0)\mathrm{e}^{-\lambda T}\mathbb{P}\left[\min_{s\le T} x+cs+\sigma W_s\le 0\right].$$

By (4.2), we get $\lim_{T\to 0+}\frac{\partial G_A}{\partial T}(x,T)=0$. We have obtained the upper bound in (4.4). This completes the proof. □

Corollary 4.1 *Assume the conditions of Theorem 4.1(c) hold . In the special case that* $\psi\equiv 1$ *and* $\mathbb{P}[V_0/K_0\le 1/(1+U_1)]=\int_x^\infty dF(y)=1$*, we have* $\lim_{T\to 0+} s(V_0,T)$ *exists and is equal to* λ*.*

Remark 1. For $\psi \equiv 1$, we have the extreme case that the bond pays nothing at maturity if default occurs. Corollary 4.1. gives that if further the jump of firm value only focuses on eroding the initial log-survival ratio $\log(V_0/K_0)$ then we have strictly positive credit spread $\lambda > 0$ for bond with nearly zero maturity.

□

Remark 2. We compare results in [24] with conclusions of Theorem 4.1. In [24], Zhou considered normally distributed jump sizes and a special form of writedown function. Then, by simulation, he found the followings.(Below $\epsilon > 0$ is very small.)

- In Figure 1 of [24], the greater the volatility of jump size distribution is, the greater the credit spread $s(V_0, \epsilon)$ is. Compare this with Theorem 4.1(c).
- In Figure 2 and Figure 7 of [24], although there is possibility of jumps, no matter how large the intensity λ is and how much the jump distribution may concentrate on a level above the initial log-survival ratio $x = \log(V_0/K_0)$, the instantaneous default probability $\lim_{T\to 0+} \mathbb{P}_x[\tau(T) \le T] = 0$. We proved this in Theorem 4.1(b) for general jump distributions and general bounded nonnegative writedown functions.
- In Figure 6 of [24], the larger the initial firm value V_0 is, the smaller the credit spread $s(V_0, \epsilon)$ is. Compare this with Theorem 4.1(c).
- In Figure 5 of [24], the smaller the intensity λ is, the smaller $s(V_0, \epsilon)$ is. Compare this with Theorem 4.1. Note that for every small jump intensity, Theorem 4.1(c) gives a good estimate of $s(V_0, \epsilon)$.
- In Figure 8 of [24], large V_0 tends to imply small expected writedown $\Phi(\log(V_0/K_0), T)$. Compare this with Theorem 4.1(a).

On the other hand, although the modeling in [9] is different from ours in some aspects, it is interesting to compare their numerical results with Theorem 4.1. For related results, see for example Section 2.4 in [12].

□

5 Infinite Series Expression for Bond Price

Now, consider the space $E_T \triangleq \mathbb{R}_+ \times [0, T]$, $T > 0$. Take $E = \mathbb{R}_+ \times [0, \infty)$ for notational convenience. The positive operator L on $\mathcal{B}(E_T)_b$ is defined by the formula,

$$Lf(x,t) = \int_0^T \mathrm{d}F_{J_1}(t) \int \mathrm{d}F(y) \int_{y^+}^{\infty} \mathrm{d}w f(w-y, T-t) H(x,w,t).$$

where H is defined in (3.14). Here $f \in \mathcal{B}(E_T)_b$ is a bounded $\mathcal{B}(E_T)$-measurable function. Then by Proposition 3.1. and 3.2., we can write Φ as

$$\Phi(x,t) = G(x,t) + L\Phi(x,t), \forall (x,t) \in E. \tag{5.1}$$

where

$$G \triangleq G_A + G_B + G_C \tag{5.2}$$

Now, after n-th iteration of (5.1), we have

$$\Phi(x,t) = \sum_{k=0}^{n} L^k G(x,t) + L^{n+1}\Phi(x,t) \tag{5.3}$$

where $L^0 = Id$ and $L^{k+1}f = L(L^k f)$. From the definition of L, one sees that

$$Lf(x,t) = \mathbb{E}_x \left[f(X_{J_1}, t - J_1)\mathbf{1}\left(J_1 \le t, \min_{0\le s\le J_1} X_s > 0 \right) \right].$$

In general, after nth iteration, we have the following result.

Lemma 5.1 *Let f be a nonnegative $\mathcal{B}(E_T)-$measurable function. Then for any $n \in \mathbb{N}$,*

$$L^n f(x,t) = \mathbb{E}_x \left[f(X_{J_n}, t - J_n)\mathbf{1}\left(J_n \le t, \min_{0\le s\le J_n} X_s > 0 \right) \right]. \tag{5.4}$$

Proof The proof proceeds with induction. We already have the case $n = 1$ by the definition of L. Assume that for $n = k$ the conclusion of the lemma holds. Then for $n = k+1$,

$$\begin{aligned} L^{k+1} f(x,t) =& \mathbb{E}_x \left[L^k f(X_{J_1}, t - J_1)\mathbf{1}\left(J_1 \le t, \min_{0\le s\le J_1} X_s > 0 \right) \right] \\ =& \mathbb{E}_x \left[\mathbb{E}_{X_{J_1}} \left[f(X_{J_k}, v - J_k)\mathbf{1}\left(J_k \le t, \min_{0\le s\le J_k} X_s > 0 \right) \right] \Bigg|_{v=t-J_1} \mathbf{1}\left(J_1 \le t, \min_{0\le s\le J_1} X_s > 0 \right) \right], \end{aligned}$$

where we have applied the case of (5.4) for $n = k$ in the last line. On the other hand, by Strong Markov Property of X, we have

$$\begin{aligned} & \mathbb{E}_x \left[f(X_{J_{k+1}}, t - J_{k+1})\mathbf{1}\left(J_{k+1} \le t, \min_{0\le s\le J_{k+1}} X_s > 0 \right) \right] \\ =& \mathbb{E}_x \left[\mathbb{E}_x \left[f(X_{J_{k+1}}, t - J_{k+1})\mathbf{1}\left(J_{k+1} \le t, \min_{J_1\le s\le J_{k+1}} X_s > 0 \right) \Bigg| \mathcal{F}_{J_1} \right] \mathbf{1}\left(J_1 \le t, \min_{0\le s\le J_1} X_s > 0 \right) \right] \\ =& \mathbb{E}_x \left[\mathbb{E}_{X_{J_1}} \left[f(X_{J_k}, v - J_k)\mathbf{1}\left(J_k \le t, \min_{0\le s\le J_k} X_s > 0 \right) \right] \Bigg|_{v=t-J_1} \mathbf{1}\left(J_1 \le t, \min_{0\le s\le J_1} X_s > 0 \right) \right] \end{aligned}$$

Hence, we have proved that (5.4) holds for $n = k+1$. By induction hypothesis, we have proved the lemma. □

Since we cannot have infinitely many jumps on a compact interval, it is not difficult to see the following result.

Proposition 5.1 *For every $T > 0$ and $f \in \mathcal{B}(E_T)_b$, $L^n f$ converges to 0 as $n \to \infty$ uniformly on E_T.*

Proof It suffices to consider nonnegative f since we can apply the result to f^+ and f^- and then combine the results on $f^\pm$ for f. Note that by Lemma 3.1, for each $n \in \mathbb{N}$ and $(x,t) \in E_T$, we have

$$\begin{aligned} 0 \le L^n(x,t) &= \mathbb{E}_x \left[f(X_{J_n}, t - J_n)\mathbf{1}\left(J_n \le t, \min_{0\le s\le J_n} X_s > 0 \right) \right] \\ &\le \|f\|_\infty \mathbb{P}[J_n \le t] \le \|f\|_\infty \mathbb{P}[J_n \le T]. \end{aligned}$$

Since J_n is the $n-$th jump time of the compound Poisson process, we have

$$\mathbb{P}[J_n \le T] = \sum_{m=n}^{\infty} \mathrm{e}^{-\lambda T} \frac{(\lambda T)^m}{m!}.$$

This implies that $L^n f$ converges to zero uniformly on E_T as $n \to \infty$. $\square$

From this, we obtain from (5.3) that for every bounded ψ,

$$\Phi(x,T) = \lim_{n\to\infty} \left(\sum_{k=0}^{n} L^k G(x,T) + L^{n+1}\Phi(x,T) \right) = \sum_{k=0}^{\infty} L^k G(x,T), \tag{5.5}$$

where G is given as in (5.2). Write $x = \log(V_0/K_0)$. By (3.4), we get the following main result.

Theorem 5.1 *For the general nonnegative bounded writedown function ψ, the bond price of (3.4)is given by the formula*

$$D(V_0,T) = \mathrm{e}^{-rT} - \mathrm{e}^{-rT} \sum_{m=0}^{\infty} L^m G(\log(V_0/K_0), T), \tag{5.6}$$

where $G = G_A + G_B + G_C$ and G_A, G_B, and G_C are give as in (3.11), (3.12) and (3.13) respectively. Moreover the series converges uniformly on E_T.

Remark. In our setting, even if there is default prior to maturity, the remaining value of one dollar is paid until maturity date T. If one considers the case in which the remaining value is paid immediately at the time of default for which the discount factor to be used is $\mathrm{e}^{-r\tau}$, then a formula similar to (5.6) can be derived with exactly the same scheme.

$\square$

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