The Sequential Compound Option Pricing with Random Interest Rate and Applications to Project Valuation

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Abstract

This paper proposes the pricing formula of sequential compound options (SCOs) with random interest rate and the applications call Milestone Project Valuation (MPV). Most compound options in literatures are 2-fold with constant parameters through time. The multi-fold compound options are just sequential compound CALL options. The multi-fold sequential compound options proposed in this study are compound option on (compound) option with random interest rate and allow call/put alteration. Besides, the parameters can vary in different folds and make the model more flexible. The SCOs can enhance and broaden the usages of compound option in real option and financial derivative fields, including MPV. The projects that set some critical milestones, which should be achieved sequentially, are called milestone projects. This study propose the milestone project valuation by SCOs with random interest rate.

Keywords: sequential compound option, project valuation, real option, random interest rate, option pricing

1. Introduction

Compound options, the options with options as underlying, are one of the important financial innovations. The fold number of a compound option counts how many option layers tacking directly on other underlying options. Original compound options are proposed by Geske (1979) with 2-fold. A specific multi-fold compound option formula is revealed by Carr (1988) while the sequential compound call (SCC) is proved by Thomassen & Van Wouwe (2001) and Chen (2003). Chen (2002) and Lajeri-Chaherli (2002) prove the 2-fold compound options through risk-neutral method simultaneously. Elettra & Rossella (2003) generalized the 2 fold compound calls by time-dependent parameters.

Many financial applications extending from the compound option theory are widely employed. The seminal study by Geske and Johnson (1984a) derived the analytic American put option under the inspiration of compound option, while Carr (1988) presented the sequential exchange options formula. Corporate debt (Geske & Johnson, 1984b; Chen, 2003), chooser options (Rubinstein, 1992), capletions and floortions (the options of the interest rate options) (Musiela & Rutkowski, 1998) are also priced by compound options.

Besides the financial derivative pricing, the compound option theory is also used widely in the real option field originated by Trigeorgis (1993). Several examples include capital budget decision (Duan et al., 2003), project valuation of new drugs (Cassimon et al., 2004), production and inventory (Cortazar & Schwartz, 1993) . Compound options are very common and versatile in many real-world cases (Copeland and Antikarov, 2003).

However, the sophisticated structure of the derivative pricing and the wide deployment in the real option field make the current compound option methodology insufficient. The 2-fold compound options are not enough for the block-building financial innovations whereas the multi-fold compound options focus on the sequential compound calls only. Although Remer et al. (2001, p.97) mention that "…, *in practice, different project phases often have different risks that warrant different discount rates"*, but the important feature of fold-dependent parameters are seldom taken into consideration by compound options methodology.

This paper uses vanilla European options as building blocks and extends the compound option theory to multi-fold sequential compound options with random interest rate as well as (SCOs) alternating puts and calls (see Table 1). The SCOs are (compound) options on compound options, where the option features of different folds could be assigned arbitrarily as call or put. The SCOs presented in this paper allow the parameters (such as volatility) to vary in different folds. The random interest rate model derived by the forward measure enable the long-term SCOs more realistic.

The explicit valuation formula and sensitivity analysis of SCOs are proved by the risk-neutral method in this study. Comparing with the P.D.E. method, there is more financial intuition coinciding with the risk-neutral SCOs derivation.

Table 1 Evolutions of Compound Option Theory

Reference	Fold Number	Approach	Generalization	
			Put-Call alternating	fold-dependent parameters
Geske (1979)	2	PDE	Put/Call	No
Elettra & Rossella (2003)	2	PDE	Call	Yes
Chen (2002); Lajeri-Chaherli (2002)	2	Risk-neutral	Put/Call	No
Carr(1988), Chen (2003)	Multiple	Risk-neutral	Call	No
Thomassen & Van Wouwe (2001)	Multiple	PDE	Call	No
This Paper	Multiple	Risk-neutral	Put/Call	Yes

The multi-fold SCOs alternating puts/calls with fold-dependent parameters can enhance the compound option application, especially in real option fields. The real world cases may often be multiple interacting options containing different option types (Trigeorgis,1993), such as expansion, contraction, shutting down, abandon, switch and or growth. The interaction between different types of options could be evaluated by the SCOs. The SCOs proposed in this study make the exotic multiple interacting option valuation possible. Also, the financial derivative pricing, such as exotic chooser options and capletions, can also employ the SCOs.

The applications of SCOs call Milestone Projection Valuation (MPV) is proposed in this paper. The projects that set some critical milestones, which should be achieved sequentially, are called milestone projects. The milestone projects would fail if any one of the serial milestones is not completed. The MPV method is designed for multi-stage project based on the results of SCO. Each milestone completion has the choice to enter the next stage or not, hence the sequential project milestone could be viewed as the sequential compound CALL options.

This paper is arranged as the following: section 2 presents the SCOs pricing formula. Section 3 exhibits the MPV method. The paper ends with the conclusion.

2. Sequential Compound Options with Random Interest Rate

This section defines the notations and derives the multi-fold SCOs with interest rate through the forward measure method. The SCOs, composed of European options, are the (compound) option on compound options, where the option features of different folds could be assigned arbitrarily as either call or put. A foundation theorem constructing the k-variate normal integration by (k–1)-variate for the derivation is stated first.

Denote the correlation matrix $\mathbf{Q}_k := [Q_{\{k\},g,h}]_{k\times k}$, where $Q_{\{k\},g,h}$ is the symmetric (g, h) element of the $\mathbf{Q}_k$, $\forall 1 \le g \le h \le k$. Similarly, $d_{\{k\},g}$ is the g-th element of the vector $\left[d_{\{k\},g}\right]_{k\times 1}$. $\left([Q_{\{k\},g,h}]_{k\times k}\right)^{(-i,-j)}$ is the (k–1) by (k–1) matrix which exclude the i row and the j column of $[Q_{\{k\},g,h}]_{k\times k}$. Denote the function $f(z)=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$. The k-variate normal integral with upper bound limit vector $\left[d_{\{k\},g}\right]_{k\times 1}$ and correlation matrix $\mathbf{Q}_k$ is charactered as

$$\mathbf{N}_k\left\{\left[d_{\{k\},g}\right]_{k\times 1};\mathbf{Q}_k\right\}=\int_{-\infty}^{d_{\{k\},1}}\int_{-\infty}^{d_{\{k\},2}}\cdots\int_{-\infty}^{d_{\{k\},k}}\frac{1}{(2\pi)^{\frac{k}{2}}\sqrt{|\mathbf{Q}_k|}}e^{-\frac{1}{2}\mathbf{Z}'\mathbf{Q}_k^{-1}\mathbf{Z}}\mathrm{d}z_k\mathrm{d}z_{k-1}\cdots\mathrm{d}z_1,$$

where $\mathbf{Z}'=[z_1, z_2, \cdots, z_k]$, $\mathbf{N}_0 \equiv 1$. The following theorem states related contributions about the multivariate normal integrals.

Theorem 1

(a) **The relationship between (k-1) and k–variate normal integrals** (Curnow & Dunnett, 1962)

$$\forall 1 \le v \le k, \mathbf{N}_k\left\{\left[d_{\{k\},g}\right]_{k\times 1};\mathbf{Q}_k\right\}=$$

$$\int_{-\infty}^{d_{\{k\},v}} f(z_v)\mathbf{N}_{k-1}\left\{\left(\left[\frac{d_{\{k\},g}-Q_{\{k\},v,g}z_v}{\sqrt{1-\left(Q_{\{k\},v,g}\right)^2}}\right]_{k\times 1}\right)^{(-v,)};\left(\left[\frac{Q_{\{k\},g,h}-Q_{\{k\},v,g}Q_{\{k\},v,h}}{\sqrt{1-\left(Q_{\{k\},v,g}\right)^2}\sqrt{1-\left(Q_{\{k\},v,h}\right)^2}}\right]_{k\times k}\right)^{(-v,-v)}\right\}\mathrm{d}z_v$$

(b) **The decomposition of a multivariate normal integral** (Schroder, 1989)

$$\mathbf{N}_k\left\{\left[d_{\{k\},g}\right]_{k\times 1};\mathbf{Q}_k\right\}=\int_{-\infty}^{d_{\{k\},v}}\mathbf{N}_{v-1}\left\{\left[\frac{d_{\{k\},g}-Q_{\{k\},g,v}z_v}{\sqrt{1-Q^2_{\{k\},g,v}}}\right]_{(v-1)\times 1};\left[\frac{Q_{\{k\},g,h}-Q_{\{k\},g,v}Q_{\{k\},h,v}}{\sqrt{1-Q^2_{\{k\},g,v}}\sqrt{1-Q^2_{\{k\},h,v}}}\right]_{(v-1)\times(v-1)}\right\}$$
$$\times\mathbf{N}_{k-v}\left\{\left[\frac{d_{\{k\},v+g}-Q_{\{k\},v,v+g}z_v}{\sqrt{1-Q^2_{\{k\},v,v+g}}}\right]_{(k-v)\times 1};\left[\frac{Q_{\{k\},v+g,v+h}-Q_{\{k\},v,v+g}Q_{\{k\},v,v+h}}{\sqrt{1-Q^2_{\{k\},v,v+g}}\sqrt{1-Q^2_{\{k\},v,v+h}}}\right]_{(k-v)\times(k-v)}\right\}f(z_v)\mathrm{d}z_v$$

where $\mathbf{Q}_k$ is the correlation matrix, $\forall 1 \le v \le k$.

In Theorem 1, (a) reveals that the k-variate normal integral can be constructed from the (k–1)-variate by adding another dimension to the upper limit vector and correlation matrix. (b) states that the specific multivariate normal integral can be partitioned as another two productions of less variate. Before applying this theorem for the sequential compound option pricing, the notations are described as follows.

Assume $T_{u-1} < T_u$, $\forall u \geq 1$. For any time interval from T_{u-1} to T_u ($u \geq 1$) with interval size τ_u, its annualized volatility of asset price of this period are σ_u^2. Assume the volatility of forward price, $\sigma(t)$, is fold-wise constant, $\sigma(t) = \sigma_i$, $\forall T_{i-1} < t \leq T_i$. Denote the asset price at time T_u as S_u. Denote the interest rate process $r(t)$, $0 \leq t \leq T_i$ and the discount process $D(t) = e^{-\int_0^t r(u)du}$.

Denote $\Psi_i^{®}(T_0)$ as the random interest rate i-fold SCO with strike K_1 starting at time T_0 and expiring at time T_1. Its underlying asset is the (i–1)-fold SCO, $\Psi_{i-1}^{®}(T_1)$, active from T_1 to T_2. Hence, the underlying SCO with fold number (i–u+1), $\Psi_{i-u+1}^{®}(T_{u-1})$, validates from T_{u-1} to T_u with strike K_u and parameters σ_u^2 under the assumption that the last fold SCO starting from T_0. The first fold option, $\Psi_1^{®}(T_{i-1})$, is a vanilla option with the asset as its underlying. It should note that fold numbers come in the reverse order.

The option feature, $\Lambda_{u,u}$, represents the call or put attribute of the (underlying) SCO with fold number (i–u+1) ranging from T_{u-1} to T_u, $\forall u \geq 1$. If the SCO of this fold is a call, $\Lambda_{u,u}$=1; otherwise, $\Lambda_{u,u}$=–1 for the put. For example, a call on a put (2-fold compound option) starting at T_0 has option features with $\Lambda_{1,1}$=1 and $\Lambda_{2,2}$=–1. Denote $\Lambda_{h,g} = \prod_{u=g}^{h} \Lambda_{u,u}, \forall 1 \leq g \leq h$, and $\Lambda_{1,0} \equiv 1$. Figure 1 shows the notations of the i-fold SCO starting from T_0.

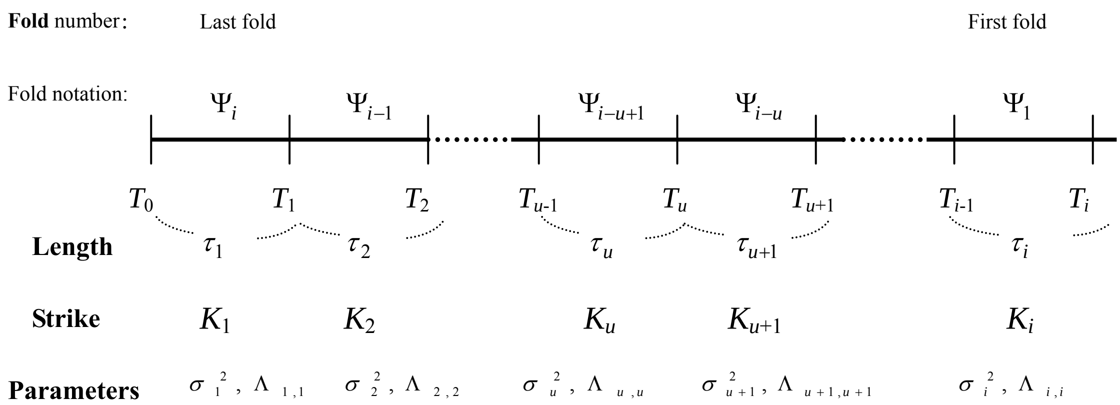

Figure 1: The Notation of the *i*-fold Sequential Compound Option

Under the same assumption of Thomassen and Van Wouwe (2001) except "parameters constant in each fold", the following theorem derives the pricing formula of the *i*-fold SCO with random interest rate alternating calls and puts by the risk-neutral method. Although the SCO presented in later section can start at any time T_u, the SCO in this theorem is starting from T_0 without loss of generality. The notation " $*v$ ", meaning "start from time T_v", is designed for time shift in sensitivity derivation. Under the above notations, denote $\Psi_i^{®}(T_0)$ as the time T_0 SCO price, which is represented in Theorem 2.

Theorem 2: Sequential compound option pricing with random interest rate

Denote

$$(1)\; a_{i,g}^{®} = \frac{\ln\left(\frac{F(0,T_g)}{S_{\#g,i}^{®}}\right) + \frac{1}{2}\sum_{u=1}^{g}\sigma_u^2\tau_u}{\sqrt{\sum_{u=v+1}^{v+g}\sigma_u^2\tau_u}},\quad \forall g \ge 1$$

$$(2)\; b_{i,g}^{®} = a_{i,g}^{®} - \sqrt{\sum_{u=1}^{g}\sigma_u^2\tau_u},\quad \forall g \ge 1$$

$$(3)\; \hat{\rho}_{g,h} = \Lambda_{h-1,g}\rho_{g,h}, \forall h > g \ge 1;\ \rho_{g,g} = 1, \forall g;\ \rho_{g,h} = \rho_{h,g}, \forall h, g;$$

$$\rho_{g,h} = \sqrt{\frac{\sum_{u=1}^{g}\sigma_u^2\tau_u}{\sum_{u=1}^{h}\sigma_u^2\tau_u}}, \forall 1 \le g < h.$$

(4) Equivalent asset price

$$\text{(EAP)}\quad S_{\#g,i}^{®} = \begin{cases} K_i, \text{for } g = i \\ \text{The stock price which makes } \Psi_{i-g}^{®}(T_g) = K_g,\ \forall 1 \le g < i \end{cases},$$ then

$$\Psi_i^{®}(T_0) = \Lambda_{i,1}S_0\mathbf{N}_i\left\{\left[\Lambda_{i,g}a_{i,g}^{®}\right]_{i\times 1};\left[\hat{\rho}_{g,h}\right]_{i\times i}\right\} - \sum_{j=1}^{i}\Lambda_{j,1}B(0,T_j)K_j\mathbf{N}_j\left\{\left[\Lambda_{i,g}b_{i,g}^{®}\right]_{j\times 1};\left[\hat{\rho}_{g,h}\right]_{j\times j}\right\} \quad \text{……(2.1)}$$

with assumptions that the EAP ($S_{\#g,i}^{®}$) exists, $\forall 1 \le g \le i$.

Pf:

This theorem is proved by induction. Let T be a fixed maturity date and $\widetilde{\mathbb{P}}^T$ be the risk-neutral measure. Define the T-forward measure $\widetilde{\mathbb{P}}^T$ by

$$\widetilde{\mathbb{P}}^T(A) = \frac{\int_A D(T)d\widetilde{\mathbb{P}}}{B(0,T)}, \text{ for all } A \in \mathcal{F}.$$

The forward price is a martingale under the forward measure. $\widetilde{W}^T(\tau)$ is the Brownian Motion used to derive the asset price under the forward measure. Hence the dynamics of the forward price $F(t, T)$ is

$$dF(t,T) = \sigma(t)F(t,T)d\widetilde{W}^T(t)$$

By Ito Lemma, $F(t,T) = F(0,T)e^{\left\{-\frac{1}{2}(\sigma(t))^2 t + \sigma(t)\widetilde{W}^T(t)\right\}}$. ……(2.2)

Take the asset price $S(t)$ as the numéraire, and the risk neutral measure of this numeraire is given by

$$\widetilde{\mathbb{P}}^S(A) = \frac{\int_A D(T)S(T)d\widetilde{\mathbb{P}}}{S(0)}, \text{ for all } A \in \mathcal{F}.$$

the asset price is identically 1 and the zero coupon bond is

$$\frac{B(t,T)}{S(t)} = \frac{1}{F(t,T)}, 0 \le t \le T.$$

By Ito Lemma,

$$d\left[\frac{1}{F(t,T)}\right] = \frac{-\sigma(t)\left[d\widetilde{W}^T - \sigma_1 dt\right]}{F(t,T)}.$$

By the Change of Risk-neutral Measure Theorem (Shreve, 2004), $\frac{1}{F(t,T)}$ is a martingale under $\widetilde{\mathbb{P}}^S$. Hence, $d\left[\frac{1}{F(t,T)}\right] = \frac{-\sigma(t)d\widetilde{W}^S}{F(t,T)}$ with the definition

$$d\widetilde{W}^S = d\widetilde{W}^T - \sigma(t)dt \qquad \text{…... (2.3)}$$

By Ito Lemma again, $d\ln\left[\frac{1}{F(t,T)}\right] = -\frac{1}{2}\sigma^2(t)dt - \sigma(t)d\widetilde{W}^S$. Thus,

$$\frac{1}{F(t,T_1)} = \frac{1}{F(0,T_1)}e^{\left\{-\frac{1}{2}\sigma_1^2 t - \sigma_1\widetilde{W}^s(t)\right\}},$$

$$\frac{1}{F(t,T)} = \frac{1}{F(0,T)}e^{\left\{-\frac{1}{2}(\sigma(t))^2 t + \sigma(t)\widetilde{W}^S(t)\right\}}.$$

The (2.1) is true for i=1. For the case $\Lambda_{1,1}$=+1 is exhibited in Shreve (2004) and the other case, $\Lambda_{1,1}$=−1, can be proved by the similar way.

Assume the (2.1) is true for the i-fold compound option $\Psi_i^{®}(T_0)$, it is showed that the (2.1) is also true for the $i+1$-fold compound option, for any $\Lambda_{g,g}$, $1 \le g \le i+1$.

Because the underlying asset of $\Psi_{i+1}^{®}(T_0)$ is $\Psi_i^{®}(T_1)$, instead of $\Psi_i^{®}(T_0)$, the start time of the i-fold compound option is shift from T_0 to T_1. All the notations of the i-fold compound option are changed simultaneously according to the time shift.

Hence $\Psi_i^{®}(T_1) = \Lambda_{i+1,2} S_1 \mathbf{N}_i \left\{ \left[\Lambda_{i+1,g+1} a_{i,g,*1}^{®} \right]_{i \times 1}; \left[\hat{\rho}_{g,h,*1} \right]_{i \times i} \right\}$

$$-\sum_{j=1}^{i} \Lambda_{j+1,2} B(T_1, T_{j+1}) K_{j+1} \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b_{i,g,*1}^{®} \right]_{j \times 1}; \left[\hat{\rho}_{g,h,*1} \right]_{j \times j} \right\} \qquad ......(2.4)$$

At the maturity time T_1 of the i+1-fold compound option, the price $\Psi_{i+1}^{®}(T_1) = \max[\Lambda_{1,1} \Psi_i^{®}(T_1) - \Lambda_{1,1} K_1]$; at the start time T_0, the option price $\Psi_{i+1}^{®}(T_0) = \widetilde{E}\{D(T_1) \max[\Lambda_{1,1} \Psi_i^{®}(T_1) - \Lambda_{1,1} K_1] \| \mathscr{F}_0\}$, according to the fundamental theory of asset pricing (Baxter and Runie, 1996), where $\widetilde{E}$ is the expectation operator under risk-neutral measure and $\mathscr{F}_0$ denotes the information available at time T_0 from the asset price.

$$\Psi_{i+1}^{®}(T_0) = \widetilde{E}\{D(T_1) \Lambda_{1,1} \Psi_i^{®}(T_1) \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}}\} - \widetilde{E}\{D(T_1) \Lambda_{1,1} K_1 \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}}\}$$

$$= \Lambda_{1,1} \widetilde{E}\left\{ \Lambda_{i+1,2} S_1 \mathbf{N}_i \left\{ \left[\Lambda_{i+1,g+1} a_{i,g,*1}^{®} \right]_{i \times 1}; \left[\hat{\rho}_{g,h,*1} \right]_{i \times i} \right\} \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}} \right\}$$

$$-\sum_{j=1}^{i} \Lambda_{1,1} \widetilde{E}\left\{ D(T_1) \Lambda_{j+1,2} B(T_1, T_{j+1}) K_{j+1} \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b_{i,g,*1}^{®} \right]_{j \times 1}; \left[\hat{\rho}_{g,h,*1} \right]_{j \times j} \right\} \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}} \right\}$$

$$-\Lambda_{1,1} B(0, T_1) K_1 \widetilde{E}\left\{ \frac{D(T_1)}{B(0, T_1)} \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}} \right\}$$

$$\equiv \widetilde{\Psi}_{i+1,1}^{®} - \widetilde{\Psi}_{i+1,2}^{®} - \widetilde{\Psi}_{i+1,3}^{®}$$

$$\Psi_{i+1,1}^{®}(T_0) = \Lambda_{i+1,1} S_0 \widetilde{E}\left\{ \frac{D(T_1) S_1}{S_0} \mathbf{N}_i \left\{ \left[\Lambda_{i+1,g+1} a_{i,g,*1}^{®} \right]_{i \times 1}; \left[\hat{\rho}_{g,h,*1} \right]_{i \times i} \right\} \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}} \right\}$$

$$= \Lambda_{i+1,1} S_0 \widetilde{E}^S \left\{ \mathbf{N}_i \left\{ \left[\Lambda_{i+1,g+1} a_{i,g,*1}^{®} \right]_{i \times 1}; \left[\hat{\rho}_{g,h,*1} \right]_{i \times i} \right\} \mathbb{1}_{\{\Lambda_{1,1}\Psi_i^{®}(T_1) > \Lambda_{1,1} K_1\}} \right\},$$

where $\widetilde{E}^S$ is the expectation operator under $\widetilde{\boldsymbol{P}}^S$. According to (2.2) and (2.3),

$F(T_1,T_{g+1})=F(0,T_{g+1})e^{\left\{\frac{1}{2}\sigma_1^2\tau_1+\sigma_1\widetilde{W}^S(\tau_1)\right\}}$. The following shows that applying the last result to $\Psi^{®}_{i+1,1}(T_0)$.

$$\widetilde{a}^{®}_{i,g,*1}=\frac{\ln\left[\dfrac{F(0,T_{g+1})}{S^{®}_{\#g+1,i+1}}\right]+\dfrac{1}{2}\sum_{u=1}^{g+1}\sigma_u^2\tau_u+\sigma_1\widetilde{W}^S(\tau_1)}{\sqrt{\sum_{u=2}^{g+1}\sigma_u^2\tau_u}}=\frac{a^{®}_{i+1,g+1}+z_{12}\rho_{1,g+1}}{\sqrt{1-\rho_{1,g+1}^2}}\equiv\overline{a}^{®}_{i,g,*1},\forall 1\le g\le i\ .$$

The Brownian Motion $\widetilde{W}^S(\tau_1)$ is replaced by $z_{12}\sqrt{\tau_1}$, where z_{12} is the standard normal random variable. Then,

$\Psi^{®}_{i+1,1}(T_0)=\Lambda_{i+1,1}S_0\Lambda_{i+1,1}\int_{-a^{®}_{i+1,1}}^{\Lambda_{i+1,1}\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z_{12}^2}\mathbf{N}_i\left\{\left[\Lambda_{i+1,g+1}\overline{a}^{®}_{i,g,*1}\right]_{i\times 1};\left[\hat{\rho}_{g,h,*1}\right]_{i\times i}\right\}dz_{12}$. Note that the lower limit of the integration is also changed by (2.3).

Denote $z_{13}=-\Lambda_{i+1,1}z_{12}$, hence

$$\begin{aligned}\widetilde{\Psi}^{®}_{i+1,1}&=\Lambda_{i+1,1}S_0\int_{-\infty}^{\Lambda_{i+1,1}a^{®}_{i+1,1}}\frac{e^{-\frac{1}{2}z_{13}^2}}{\sqrt{2\pi}}\mathbf{N}_i\left\{\left[\frac{\Lambda_{i+1,g+1}a^{®}_{i+1,g+1}-\Lambda_{g,1}\rho_{1,g+1}z_{13}}{\sqrt{1-\left(\Lambda_{g,1}\rho_{1,g+1}\right)^2}}\right]_{i\times 1};\left[\hat{\rho}_{g,h,*1}\right]_{i\times i}\right\}dz_{13}\\&=\Lambda_{i+1,1}S_0\mathbf{N}_{i+1}\left\{\left[\Lambda_{i+1,g}a^{®}_{i+1,g}\right]_{(i+1)\times 1};\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}\right\}\end{aligned}$$

The last equation is derived by Theorem 1 (a) and the following is to exhibit that $\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}=\left[\rho_{g,h}\right]_{(i+1)\times(i+1)}$.

According to Theorem 1, $H_{0,1,1}=1$; $H_{0,1,g}=\Lambda_{h-1,1}\rho_{1,h},\forall 2\le g\le i+1$; $H_{0,g,h}=H_{0,h,g}$; $H_{0,g,g}=1$, $\forall 2\le g\le i+1$. $\forall 2\le g<h\le i+1$,

$$\begin{aligned}H_{0,g,h}&=\Lambda_{g-1,1}\rho_{1,g}\Lambda_{h-1,1}\rho_{1,h}+\sqrt{1-(\Lambda_{g-1,1}\rho_{1,g})^2}\sqrt{1-(\Lambda_{h-1,1}\rho_{1,h})^2}\,\hat{\rho}_{g-1,h-1,*1}\\&=\Lambda_{h-1,g}\frac{\sqrt{\sum_{u=1}^{g}\sigma_u^2\tau_u}}{\sqrt{\sum_{u=1}^{h}\sigma_u^2\tau_u}}=\Lambda_{h-1,g}\rho_{g,h}=\hat{\rho}_{g,h}\ .\end{aligned}$$

According to the above statements, $\left[H_{0,g,h}\right]_{(i+1)\times(i+1)}=\left[\rho_{g,h}\right]_{(i+1)\times(i+1)}$ and $\widetilde{\Psi}^{®}_{i+1,1}=\Lambda_{i+1,1}S_0\mathbf{N}_{i+1}\left\{\left[\Lambda_{i+1,g}a^{®}_{i+1,g}\right]_{(i+1)\times 1};\left[\hat{\rho}_{g,h}\right]_{(i+1)\times(i+1)}\right\}$.

The $\widetilde{\Psi}^{®}_{i+1,2}$ and $\widetilde{\Psi}^{®}_{i+1,3}$ can be derived under the T-forward measure $\widetilde{\boldsymbol{P}}^T$, which has the expectation operator $\widetilde{E}^T$.

$$\widetilde{\Psi}^{®}_{i+1,2} = \sum_{j=1}^{i} \Lambda_{j+1,1} K_{j+1} B(T_0, T_{j+1}) \widetilde{E} \left\{ \frac{D(T_1)}{B(0,T_1)} \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b^{®}_{i,g,*1} \right]_{j\times 1} ; \left[\hat{\rho}_{g,h,*1} \right]_{j\times j} \right\} \mathbb{1}_{\left\{ \Lambda_{1,1} \Psi^{®}_{i}(T_1) > \Lambda_{1,1} K_1 \right\}} \right\}$$

$$= \sum_{j=1}^{i} \Lambda_{j+1,1} K_{j+1} B(T_0, T_{j+1}) \widetilde{E}^{T} \left\{ \mathbf{N}_j \left\{ \left[\Lambda_{i+1,g+1} b^{®}_{i,g,*1} \right]_{j\times 1} ; \left[\hat{\rho}_{g,h,*1} \right]_{j\times j} \right\} \mathbb{1}_{\left\{ \Lambda_{1,1} \Psi^{®}_{i}(T_1) > \Lambda_{1,1} K_1 \right\}} \right\}$$

Substitute the (2.2) into the above equation and apply the similar way of $\widetilde{\Psi}^{®}_{i+1,1}$, it can be derived that $\widetilde{\Psi}^{®}_{i+1,2} = \sum_{j=2}^{i+1} \Lambda_{j,1} B(0,T_j) K_j \mathbf{N}_j \left\{ \left[\Lambda_{j,g} b^{®}_{j,g} \right]_{j\times 1} ; \left[\hat{\rho}_{g,h} \right]_{j\times j} \right\}$.

$\widetilde{\Psi}^{®}_{i+1,3}$ can be derived by the same method. $\widetilde{\Psi}^{®}_{i+1,3} = \Lambda_{1,1} B(0,T_1) K_1 \mathbf{N}_1 \left\{ \Lambda_{i+1,1} b^{®}_{i+1,1} \right\}$. Consequently, $\Psi^{®}_{i+1}(T_0)$ is proved. **Q.E.D.**

According to (2.1), the *i*–fold SCO can be priced by the asset price minus strike prices of the *i* different underlying with weights. The weights are multiplied by three factors, including the cumulative option features, the discount factors and the in-the-money probabilities. The cumulative option feature is the synthetic option feature from the standing fold to the last fold, whereas the discount factor is deduction made by either depression or interest rate. The in-the money probabilities are accessed for different fold SCOs under different probability measures by the multivariate normal integrations. The $a_{i,g}$ and $b_{i,g}$ in the integration are similar to the conventional "d_1" and "d_2" in vanilla options. The correlation matrices of SCOs are similar to the sequential compound calls except the sign change due to the cumulative option features. For the 3 weighting factors, the impact to the pricing formula by parameters of the last fold is the strongest.

The SCOs formulas are close to those of vanilla options, 2–fold compound options and sequential compound calls, which can be regarded as special cases of SCOs. The main difference between SCOs and sequential compound calls (Thomassen & Van Wouwe, 2001) lies in alternating calls and puts of SCOs, which are manifested by sign changes of cumulative option features, $\Lambda_{h,g}$, $\forall 1 \leq g \leq h$. Moreover, the parameters varying with folds in this study also make the integrated volatility different to the vanilla options and Thomassen & Van Wouwe (2001). Thus the SCOs setting all $\Lambda_{h,g}$ as +1 becomes the sequential compound call.

3. The Milestone Project Valuation (MPV)

This section proposes the Milestone Projection Valuation (MPV) method for the multi-stage projects. The projects that set some critical milestones which should be achieved sequentially are call milestone projects (see Figure 2 for example). The milestone projects are failed if any one of the serial milestones is not completed. The milestone projects are very common in real situation, including R&D management, manufactures, etc. Originally, the milestone projects are valuated by the methods including the net prevent values (NPV) and decision trees. The NPV method valuates a project under a rigorous assumption that all future cash flows are certain. Obviously, the uncertainty is ignoring in the NPV method and results in symmetric underestimates. Recently, the popular real option approach is applied for flexible consideration and reasonable explanation. Under the framework of financial option theory, the real option approach decomposes the project valuation as several parameters, including the present value, costs, time to maturity, value uncertainty (volatility) and interest rate. Most of the existing real option studies for the multi-stage milestone project valuations use one-fold options, while the others apply multi-fold options under the assumption of constant parameter through whole the processes (Casimon et al., 2004). However, the parameters often change due to the milestone completion and the project values will be misestimated if parameters are assumed constant through all the time. The one-fold real option approach for multi-stage project is even inadequate.

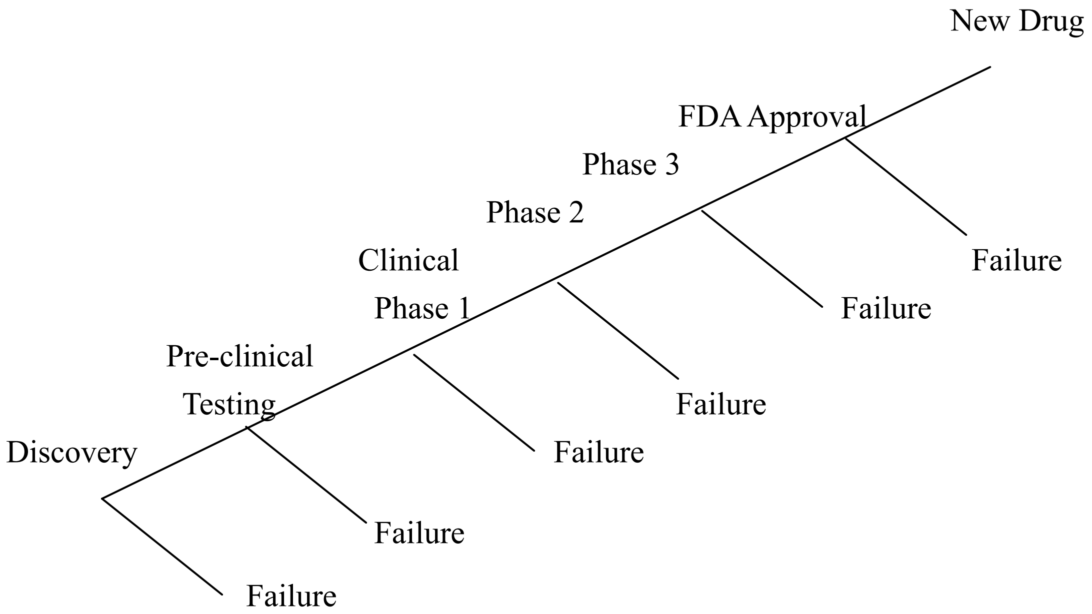

Figure 2 A Milestone Project: the New Drug Development (NDA)

Based on the results of SCOs (2.1), this paper proposes a method called Milestone Projection Valuation (MPV) for multi-stage project valuation. Each milestone completion has the choice to enter the next stage or not, and the sequential project milestone can be viewed as the sequential compound CALL options. The MPV method adopts the results of SCOs and the project is regarded as the corresponding stock in SCOs. Under the same denotations as **Theorem 2**, the MPV valuation formula is expressed as ®

$$MPV_i^{®}(T_0) = S_0\mathbf{N}_i\left\{\left[a_{i,g}^{®}\right]_{i\times 1};\left[\rho_{g,h}\right]_{i\times i}\right\} - \sum_{j=1}^{i} B(0,T_j)K_j\mathbf{N}_j\left\{\left[b_{i,g}^{®}\right]_{j\times 1};\left[\rho_{g,h}\right]_{j\times j}\right\}$$

......(3.1)

, where the strikes represent the cost at different stages; the volatilities come from the project value fluctuation and the dividend rates are replaced as the depression rates. The option features ($\Lambda_{i,g}$) equal one (for any i, g) due to the underlying compound calls, hence disappear in the MPV pricing formula.

Compared with the literatures, the MPV not only applies the multi-fold compound option theory, but also allows the parameters piece-constant varying with the distinct stages. The different parameters of different stages can adapt to the change of project nature after the milestone completion. More phenomena can be discovered from the parameter comparisons. Under the MPV model, the implicit "valuation experience" is decomposed as parameters.

The new drug developments (NDAs) may be the most famous and significant milestone projects. Under the consideration of human health, the NDAs are the well-regulated including the stages of pre-clinical trial, phase 1, phase 2, phase 3 and approval phase. Each stage has a definitive milestone. The time- and cost-consuming NDAs are the cores of the pharmaceutical companies because the R&D results from NDAs dominate their future! The MPV model can enhance the NDAs valuation under a more reasonable framework and improve the R&D management of these companies.

4. Conclusion

The puts/calls-alternating sequential compound options (SCOs) with random interest rate and fold-wise constant parameters are proposed in this study. Based on the results, the Milestone Projection Valuation (MPV) method is proposed for multi-stage project valuation.

Traditional compound options are just either puts/calls-alternating 2-fold compound options or multi-fold sequential compound call without puts/calls-alternating. Seldom fold-wise differences nor the random interest rate are taken into consideration. The SCOs with random interest rate presented in this paper have the following specialties. First of all, the multi-fold SCOs with arbitrary fold feature assignments as puts or calls can enhance the compound options usage far beyond the traditional sequential compound calls. Second, the parameters (interest rate, volatility) often vary with time or folds due to different characteristics. The presented SCOs formula enables random interest rate and volatility change fold-wise to capture the "sequential" features. The third is that the arbitrary fold number of SCOs can be formed.

The SCOs not only generalized the contributions of Black-Scholes (1973), Geske (1979) and Thomassen & Van Wouwe (2001) to put/call alternating multi-fold compound options, but can be evaluated by linear combination of the stock and strike prices weighted by different variate normal integrations. Corresponding to intuitions, the SCOs seem as the multi-dimensional options extending from Black-Scholes (1973) and Geske (1979). The risk-neutral method enriches the SCOs pricing formula derivation with financial implications.

The SCOs can enhance and broaden the usages of compound option in real option and financial derivative fields. The multiple interacting options incorporating different type real options sophisticatedly can be evaluated by aggregation of various SCOs. The milestone projects, deciding whether the projects are terminated or not by the period milestone achievement, also can be evaluated by the SCOs with random interest rate. Comparing with the constant volatility in Casimon et al. (2004), the volatilities and interest rates estimated for different periods make the project valuation more precise and flexible. Besides, more financial derivatives can be developed or valuated according to SCOs as the way that chooser options, capletions are priced by 2-fold compound options. The applications of SCOs with real-world cases will be probable future researches.

The MPV is designed for multi-stage project valuation. The MPV method adopts the results of SCO and the project is regarded as the corresponding stock in SCO. Compared with the literatures, the MPV not only applies the multi-fold compound option theory, but also allows the parameters piece-constant varying with the distinct stages. The different parameters of different stages can adapt to the change of project nature after the milestone completion. More phenomena can be discovered from the parameter comparisons. Under the MPV model, the implicit "valuation experience" is decomposed as parameters.

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