Complexity and the Character of Stock Returns: Empirical Evidence and A Model of Asset Prices Based Upon Complex Investor Learning

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Objectives of the Study

- Propose an alternative model of investor learning behavior in a dynamic, complex environment
- Confirm several characteristics of common stock returns that have previously been investigated in isolation
- Present empirical evidence on the ability of the model to reproduce the return behaviors documented for a sample of common stocks
  - Aside: Robustness of a system may be the result of the system developing into a complex structure

Outline

- An alternative approach for studying dynamic markets
  - Agent based computational modeling
- Return characteristics that have been documented in the literature
- An alternative model of investor learning behavior
- Empirical specification of the model
- Results from dynamic simulations
  - Sensitivity of the results to parameter variation
- Summary and conclusions

Agent-Based Computational Modeling

- **Computational study** of financial markets modeled as evolving decentralized systems of autonomous interacting agents
- Focus is on understanding **how global regularities arise from the bottom up**, through the repeated local **interactions** of autonomous agents
- Agents are modeled as **heterogeneous** entities that **interact** directly or indirectly with other agents and with the environment based on data and behavioral rules

Why an Alternative Approach?

- Traditional models involving deductive reasoning and rational expectations do not always do a good job of predicting stock price behavior
  - Knowledge of the structure of the underlying model
  - Learning the values of the parameters
- Deductive reasoning breaks down in complex environments where model structure may be changing over time due to heterogeneity of agent beliefs and perceptions and complexity of the environment
- In this case induction, constantly forming and testing new hypotheses, is necessary
  - Sensible? Can we ever know the true model?
  - The world is inherently dynamic and complex

Benchmark Sample

- 50 common stocks traded on the NYSE
  - Random selection
- 2780 daily stock returns
  - Source CRSP

Some Characteristics of Common Stock Returns

- Unconditional returns are not normally distributed
  - Jarque-Bera test (excess skewness and kurtosis)
- Heavy (Fat) Tails
  - Power-law estimates ($\alpha \cong 3$)

$$pr(r > x) \sim \frac{1}{x^{\alpha_i}}$$ (for large x) (heavy tails; large variance)

- Time series dependencies
  - Autocorrelation
  - ARCH/GARCH – type behavior
  - Long – range dependencies in squared returns
- Less well-documented
  - Return time-series continue to exhibit nonlinear dependencies after filtering for linear and ARCH-type effects

Some Basic Statistics and Tests

- Usual descriptive statistics
  - Mean, median, std. deviation, skewness, kurtosis
- Tail Index estimates $pr(r > x) \sim \frac{1}{x^{\alpha_i}}$
  - Fat tails
    - Hill estimator (max likelihood under IID data)
    - Quintos et al. estimator (accounts for GARCH)
- Time-series Dependencies
  - BDS statistic
  - Ljung-Box Q-statistic
  - ARCH LM statistic
  - Lo's V statistic (long-range dependencies)

Test Statistics

- Dependencies
  - BDS test
    - Test will detect both linear as well as nonlinear dependencies
  - Ljung-Box Q test
    - Test will detect linear dependencies (autocorrelation)
  - ARCH LM test
    - Test will detect autoregressive conditional heteroskedasticity
      - Regression of squares of estimated errors on their lags
  - Lo's V test
    - Test will detect long-run dependencies in the presence of short-run dependencies

BDS Examples

- Three cases
  - ***e*** is IID, u = N(0,1)
  - ***y*** follows an AR(1) process
  - ***z*** follows a non-linear moving average process

$$(1)\ e_t = u_t$$

$$(2)\ y_t = \mu + .6y_{t-1} + .5u_t$$

$$(3)\ z_t = u_t + .8(u_{t-1} \bullet u_{t-2})$$

  - Results
    - (1) no autocorrelation; ***BDS*** does not reject IID
    - (2) first order AC = .6; ***BDS*** rejects IID
    - (3) no autocorrelation but ***BDS*** rejects IID
- Thus, once we filter out any suspected linear relation, as well as suspected nonlinear relation, the *BDS* test allows us to test for any remaining nonlinear relation

Results for the Actual Data

Basic results

- Excess skewness and kurtosis relative to Normal Distribution benchmarks of 0 and 3
- Fat tails
- General evidence of dependencies
- Some evidence of autocorrelation
- Strong evidence of ARCH-type behavior
- Some evidence of long-range dependencies in squared series

Parameter Values for the Experiments

- Table 3
- Results by Case: Table 1, Table 2
  - Actual vs Base Case
  - Robustness of Results to Parameter Variation
- Results are relatively robust
  - A surprise?
  - Recent developments on the robustness of complex systems (www.physics.ucsb.edu/~complex/research/robustness.htm)
    - Hypothesis: Robustness is the underlying mechanism that leads to complexity
    - If we could iterate on a system to make it robust the resulting structure would appear be highly complex

Table 3

Parameter Values of the Artificial Stock Market Model

	Learning Frequency	$\bar{d}$	ρ	σ_{υ}^{2}	Risk-Free Interest Rate	λ	Pr(Comb) π	Pr(Indiv) $(1 - \pi)$	N	Number of Hypotheses (Rule Bases)
Case 1	***30***	***0.0137***	***0.5***	***0.0005***	***Actual***	***0.50***	***0.2***	***0.8***	***25***	***5***
Case 2	30	0.0137	0.5	0.0005	Actual	0.50	**0.5**	**0.5**	25	5
Case 3	30	0.0137	0.5	0.0005	Actual	0.50	**0.8**	**0.2**	25	5
Case 4	30	0.0137	**0.1**	0.0005	Actual	0.50	0.2	0.8	25	5
Case 5	30	0.0137	**0.9**	0.0005	Actual	0.50	0.2	0.8	25	5
Case 6	**10**	0.0137	0.5	0.0005	Actual	0.50	0.2	0.8	25	5
Case 7	**1000**	0.0137	0.5	0.0005	Actual	0.50	0.2	0.8	25	5
Case 8	30	0.0137	0.5	0.0005	Actual	0.50	0.2	0.8	**10**	5
Case 9	30	0.0137	0.5	0.0005	Actual	0.50	0.2	0.8	**50**	5
Case 10	30	0.0137	0.5	0.0005	Actual	0.50	0.2	0.8	25	**3**
Case 11	30	**0.0068**	0.5	0.0005	Actual	0.50	0.2	0.8	25	5
Case 12	30	**0.0041**	0.5	0.0005	Actual	0.50	0.2	0.8	25	5
Case 13	30	0.0137	0.5	**0.0003**	Actual	0.50	0.2	0.8	25	5
Case 14	30	0.0137	0.5	**0.0007**	Actual	0.50	0.2	0.8	25	5
Case 15	30	0.0137	0.5	0.0005	**+10% shift**	0.50	0.2	0.8	25	5
Case 16	30	0.0137	0.5	0.0005	**+10% shift**	0.50	0.2	0.8	25	5
Case 17	30	0.0137	0.5	0.0005	Actual	**0.10**	0.2	0.8	25	5
Case 18	30	0.0137	0.5	0.0005	Actual	**0.90**	0.2	0.8	25	5

Table 3

Note: The table lists the cases investigated in the simulation of the artificial stock market and the parameters varied across the cases. Learning Frequency τ: the number of periods between the dates on which any agent updates his hypotheses; $\bar{d}$: the mean dividend (eq. 6.1); ρ : adjustment factor for the dividend in the dividend generating process (eq. 6.1); σ_{υ}^{2} : variance of error in the dividend generating process (eq. 6.1); λ : the coefficient of risk aversion for each agent; Pr(Comb) = π: probability of Combination Experimentation; Pr(Indiv): probability of Individual Experimentation (= 1 – Pr(Comb) = (1 - π); *N*: the number of agents; Number of Rule Bases: Number of hypotheses about the future course of the (price + dividend) by each agent. Each Rule Base contains four rules used in the construction of the two parameters needed for predicting next period's (price + dividend) from information on 5 market determined variables (5 information bits) observed by all agents. Agents represented in the model employ induction and reason as if by fuzzy logic when forming their expectations. The process is modeled as a genetic-fuzzy classifier system. We use the daily 1-year T-bill rates in the secondary market for the 5,000 days ending September 5, 2001 when computing demands using equation (6.4). In the simulation, we divide this interest rates series by 365 to obtain the approximate daily interest rates. The Probability of Combination Experimentation is the probability that elements of two hypotheses will be split and combined. The Probability of Individual Experimentation is the probability that an agent will have one of his rule bases subjected to random change. When a particular rule base is selected for experimentation the probability that any individual information bit is changed equals 0.5.

Table 1

Average Values of Descriptive Statistics and Test Statistics
Standard Errors in Parentheses; Fraction of Tests Rejecting the Null Hypothesis in Square Brackets

	Actual	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8
Mean (x 100)	0.064	0.013	0.010	0.001	0.012	0.012	0.023	0.000	0.025
	(0.005)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.000)	(0.002)
Median (x 100)	0.000	-0.013	-0.003	-0.003	-0.014	-0.021	-0.035	0.000	-0.020
	0.000	(0.003)	(0.001)	(0.002)	(0.003)	(0.004)	(0.005)	(0.000)	(0.004)
Std. Dev. (x 100)	2.120	1.665	1.441	0.241	1.684	1.651	2.174	0.036	2.260
	(0.132)	(0.032)	(0.052)	(0.069)	(0.046)	(0.045)	(0.027)	(0.008)	(0.081)
Skewness	0.424	1.304	0.948	-0.023	1.561	1.249	2.244	0.080	0.801
	(0.049)	(0.202)	(0.160)	(0.180)	(0.167)	(0.147)	(0.114)	(0.157)	(0.099)
Kurtosis	10.434	26.967	32.022	129.050	32.747	24.618	29.280	146.490	13.775
	(0.551)	(4.905)	(2.996)	(13.333)	(3.468)	(3.473)	(2.123)	(22.903)	(2.443)
Hill's tail index	2.550	1.944	1.298	1.419	1.737	1.978	2.102	2.972	2.022
	(0.088)	(0.082)	(0.094)	(0.075)	(0.090)	(0.093)	(0.028)	(0.122)	(0.076)
Quintos' tail index	2.877	2.546	1.887	1.110	2.295	2.734	2.676	3.794	2.885
	(0.118)	(0.121)	(0.113)	(0.097)	(0.123)	(0.132)	(0.037)	(0.164)	(0.104)
BDS, $\varepsilon=1.5\sigma$, m=3	10.45	25.08	25.46	24.13	25.32	24.53	23.93	10.73	26.06
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
BDS, $\varepsilon=2\sigma$, m=3	9.71	23.74	24.20	23.86	23.87	22.76	22.13	12.67	23.47
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
Ljung-Box Q(5)	31.84	518.81	543.91	604.67	535.74	539.50	424.72	404.45	558.50
	[0.60]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
Ljung-Box Q(10)	40.45	530.50	563.86	630.59	556.54	557.04	438.18	412.52	574.57
	[0.64]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
ARCH-LM(1)	65.17	233.78	390.03	590.31	257.82	203.27	82.93	467.88	208.64
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
ARCH-LM(2)	72.61	242.83	416.79	732.15	279.70	216.08	86.88	615.28	220.03
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[0.97]	[1.00]	[1.00]
$V(180)$	1.17	1.43	2.15	6.59	1.55	1.45	1.10	5.28	1.28
	[0.00]	[0.17]	[0.67]	[0.97]	[0.27]	[0.20]	[0.03]	[0.80]	[0.10]
$V(270)$	1.13	1.44	2.10	7.82	1.59	1.48	1.22	6.34	1.28
	[0.02]	[0.17]	[0.67]	[0.97]	[0.27]	[0.23]	[0.00]	[0.83]	[0.10]
$V_s(180)$	0.761	0.618	0.779	1.026	0.728	0.605	0.513	0.711	0.649
	[0.62]	[0.77]	[0.60]	[0.07]	[0.73]	[0.80]	[1.00]	[0.50]	[0.73]
$V_s(270)$	0.639	0.535	0.677	0.941	0.639	0.524	0.436	0.657	0.553
	[0.88]	[0.90]	[0.73]	[0.10]	[0.87]	[0.90]	[1.00]	[0.63]	[0.77]

Note: Column labeled "Actual" presents results for sample series of daily with-dividend common stock returns for 50 common stocks traded on the NYSE. Data are from the CRSP archive files. All other columns are for the cases of the artificial stock market listed in Table 3 where each case represents a different selection of values for the parameters of the model. The Hill and Quintos et al. estimators of the tail index are described in section 3 of the text. The test statistics and the respective null hypotheses being tested for the tests BDS, Ljung-Box Q(q), ARCH-LM(q), $V(q)$ and $V_s(q)$ are described in section 4 of the text.

Table 1

(continued)

Average Values of Descriptive Statistics and Test Statistics

Standard Errors in Parentheses; Fraction of Tests Rejecting the Null Hypothesis in Square Brackets

	Case 9	Case 10	Case 11	Case 12	Case 13	Case 14	Case 15	Case 16	Case 17	Case 18
Mean (x 100)	0.008	0.024	0.013	0.014	0.012	0.011	0.013	0.012	0.015	0.122
	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.005)
Median (x 100)	-0.017	-0.021	-0.010	-0.022	-0.013	-0.014	-0.013	-0.013	0.000	-0.001
	(0.004)	(0.005)	(0.003)	(0.005)	(0.003)	(0.003)	(0.003)	(0.003)	(0.000)	(0.002)
Std. Dev. (x 100)	1.479	2.184	1.651	1.656	1.646	1.622	1.686	1.566	1.734	4.943
	(0.038)	(0.047)	(0.051)	(0.043)	(0.034)	(0.041)	(0.035)	(0.050)	(0.035)	(0.097)
Skewness	1.592	1.515	1.507	1.714	1.546	1.178	1.526	1.348	0.564	0.979
	(0.233)	(0.106)	(0.151)	(0.225)	(0.194)	(0.102)	(0.114)	(0.082)	(0.066)	(0.049)
Kurtosis	34.620	21.992	30.551	32.572	27.906	21.739	26.457	22.844	28.302	13.543
	(4.696)	(1.681)	(4.255)	(5.134)	(3.946)	(2.128)	(2.910)	(2.253)	(1.411)	(0.581)
Hill's tail index	1.872	1.881	1.784	1.927	1.924	1.874	2.009	1.918	0.900	1.779
	(0.087)	(0.041)	(0.089)	(0.079)	(0.072)	(0.085)	(0.082)	(0.083)	(0.017)	(0.026)
Quintos et al. tail index	2.494	2.544	2.412	2.607	2.595	2.520	2.707	2.620	1.609	2.046
	(0.107)	(0.046)	(0.119)	(0.106)	(0.099)	(0.118)	(0.104)	(0.116)	(0.034)	(0.022)
BDS, $\varepsilon=1.5\sigma$, m=3	24.03	24.31	25.90	25.68	24.58	24.95	25.43	25.82	24.92	25.06
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
BDS, $\varepsilon=2\sigma$, m=3	22.62	23.10	24.06	23.67	22.82	23.17	23.32	23.87	23.51	21.87
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
Ljung-Box Q(5)	524.28	482.17	530.46	502.54	497.90	531.04	498.89	529.13	575.97	424.81
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
Ljung-Box Q(10)	537.21	495.14	548.65	517.61	509.55	541.84	509.89	544.87	590.42	437.92
	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]
ARCH-LM(1)	281.04	136.04	204.66	159.57	138.63	205.36	155.99	159.94	569.88	143.15
	[0.97]	[1.00]	[1.00]	[1.00]	[0.97]	[1.00]	[0.97]	[1.00]	[1.00]	[1.00]
ARCH-LM(2)	308.69	146.18	219.27	174.78	146.40	217.42	167.47	175.19	614.35	167.64
	[0.97]	[1.00]	[1.00]	[1.00]	[1.00]	[1.00]	[0.97]	[1.00]	[1.00]	[1.00]
$V(180)$	1.73	1.11	1.56	1.30	1.32	1.55	1.43	1.23	2.05	0.89
	[0.43]	[0.03]	[0.20]	[0.07]	[0.07]	[0.20]	[0.17]	[0.10]	[0.73]	[0.30]
$V(270)$	1.79	1.15	1.60	1.34	1.35	1.60	1.46	1.27	1.93	0.81
	[0.40]	[0.07]	[0.23]	[0.07]	[0.13]	[0.27]	[0.17]	[0.07]	[0.53]	[0.53]
$V_s(180)$	0.730	0.631	0.646	0.697	0.642	0.680	0.648	0.663	0.691	0.616
	[0.63]	[0.87]	[0.80]	[0.77]	[0.77]	[0.70]	[0.83]	[0.70]	[0.77]	[0.87]
$V_s(270)$	0.642	0.540	0.564	0.604	0.554	0.587	0.560	0.576	0.599	0.534
	[0.87]	[0.93]	[0.93]	[0.87]	[0.83]	[0.83]	[0.87]	[0.80]	[0.90]	[0.97]

Note: Column labeled "Actual" presents results for sample series of daily with-dividend common stock returns for 50 common stocks traded on the NYSE. Data are from the CRSP archive files. All other columns are for the cases of the artificial stock market listed in Table 3 where each case represents a different selection of values for the parameters of the model. The Hill and Quintos et al. estimators of the tail index are described in section 3 of the text. The test statistics and the respective null hypotheses being tested for the tests BDS, Ljung-Box Q(q), ARCH-LM(q), $V(q)$ and $V_s(q)$ are described in section 4 of the text.

Filtering for Suspected Linear and Nonlinear Relations

- Base upon the results shown in Table 1 and elsewhere in the literature we filter each series by jointly estimating best fit models of the form ARMA(m,n)-TARCH(p,q) to remove short-term dependencies

Autoregressive Conditional Heteroskedasticity

- Threshold ARCH model
  - Allows for asymmetric reactions of volatility to good and bad news (Glosten, Jagannathan and Runkle,1993)

$$r_t = a_0 + \sum_{\tau=1}^{m} \phi_i r_{t-\tau} + \varepsilon_t - \sum_{\omega=1}^{n} \theta_i \varepsilon_{t-\omega}$$

$$\sigma_t^2 = \omega + \sum_{\eta=1}^{q} \alpha_i \varepsilon_{t-\eta}^2 + \gamma \varepsilon_{t-1}^2 d_{t-1} + \sum_{\varpi=1}^{p} \beta_j \sigma_{t-\varpi}^2$$

- The results presented in Table 2 show that the models fit the data well, in terms of short-term dependencies, **however the *BDS* test indicates that non-linear dependencies still exist and long-range dependencies in the squared errors remains**

Table 2

Average Values of Test Statistics

Tests of Hypotheses About Features of the Standardized Residuals of ARMA-TARCH Models Fit to Actual and Simulated Returns

	Actual	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6	Case 7	Case 8	Case 9	Case 10	Case 11	Case 12	Case 13	Case 14	Case 15	Case 16	Case 17	Case 18
Ljung-Box Q(5)	1.48	4.25	6.24	5.82	5.86	4.20	5.87	5.02	3.95	4.19	5.02	3.85	4.22	4.99	6.11	5.31	4.83	5.35	5.83
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
Ljung-Box Q(10)	2.41	8.22	11.80	11.55	10.58	9.25	10.42	11.29	6.66	9.19	11.18	9.73	10.02	10.15	10.45	9.59	8.48	10.66	10.41
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
ARCH-LM(5)	4.69	0.94	1.93	0.58	1.78	2.01	0.39	2.47	2.82	1.76	0.99	0.92	1.47	0.97	1.30	1.26	1.37	0.61	1.62
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
ARCH-LM(10)	8.55	1.53	3.55	1.11	3.15	3.23	1.17	3.79	4.43	3.01	1.86	2.05	2.39	2.59	2.16	2.32	1.82	1.43	2.99
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
BDS, ε=1.5σ, m=3	2.71	3.87	3.78	4.61	4.31	4.67	2.06	0.58	2.94	5.93	3.39	3.43	4.20	3.74	4.18	3.37	4.78	4.59	2.07
	[0.66]	[0.67]	[0.77]	[0.80]	[0.70]	[0.70]	[0.47]	[0.30]	[0.50]	[0.67]	[0.60]	[0.60]	[0.67]	[0.53]	[0.63]	[0.73]	[0.80]	[0.87]	[0.50]
BDS, ε=2σ, m=3	2.08	2.55	2.24	2.02	2.78	3.22	1.21	1.70	1.53	4.31	1.93	2.10	3.04	2.16	2.54	2.27	3.27	2.55	1.66
	[0.48]	[0.63]	[0.50]	[0.43]	[0.53]	[0.53]	[0.40]	[0.37]	[0.40]	[0.53]	[0.37]	[0.43]	[0.53]	[0.43]	[0.50]	[0.53]	[0.67]	[0.60]	[0.33]
$V(180)$	1.31	1.23	1.25	1.32	1.16	1.21	0.96	1.35	1.13	1.36	1.05	1.14	1.08	1.10	1.23	1.19	1.12	1.28	1.14
	[0.00]	[0.00]	[0.00]	[0.07]	[0.00]	[0.00]	[0.07]	[0.07]	[0.03]	[0.00]	[0.10]	[0.00]	[0.03]	[0.00]	[0.00]	[0.03]	[0.00]	[0.00]	[0.00]
$V(270)$	1.32	1.21	1.22	1.31	1.16	1.21	1.15	1.48	1.15	1.39	1.09	1.15	1.09	1.08	1.23	1.17	1.11	1.29	1.16
	[0.00]	[0.00]	[0.03]	[0.13]	[0.00]	[0.00]	[0.00]	[0.07]	[0.03]	[0.00]	[0.00]	[0.00]	[0.13]	[0.00]	[0.00]	[0.03]	[0.00]	[0.00]	[0.00]
$V_s(180)$	0.30	0.51	0.58	0.98	0.54	0.48	0.45	0.48	0.43	0.53	0.43	0.47	0.51	0.48	0.52	0.52	0.55	0.59	0.34
	[0.92]	[0.90]	[0.83]	[0.10]	[0.80]	[0.73]	[0.90]	[0.53]	[0.87]	[0.77]	[0.90]	[0.77]	[0.80]	[0.77]	[0.70]	[0.83]	[0.83]	[0.87]	[0.93]
$V_s(270)$	0.25	0.44	0.49	0.87	0.46	0.41	0.38	0.44	0.36	0.46	0.37	0.40	0.44	0.41	0.45	0.44	0.47	0.51	0.28
	[0.92]	[0.90]	[0.83]	[0.27]	[0.83]	[0.80]	[0.90]	[0.53]	[0.93]	[0.77]	[0.90]	[0.80]	[0.80]	[0.80]	[0.73]	[0.90]	[0.83]	[0.90]	[0.93]

Note: Column labeled "Actual" presents results for sample series of daily with-dividend common stock returns for 50 common stocks traded on the NYSE. Data are from the CRSP archive files. All other columns are for the cases of the artificial stock market listed in Table 3 where each case represents a different selection of values for the parameters of the model. The test statistics and the respective null hypotheses being tested for the tests BDS, Ljung-Box Q(q), ARCH-LM(q), $V(q)$ and $V_s(q)$ are described in section 4 of the text. The ARMA-TARCH model is described in section 5 of the text.

Summary of Results for Actual Return Data

- Evidence against Normal Distribution
- Fat Tails
- Autocorrelation
- Autoregressive Conditional Heteroskedasticity
- Long-range Dependencies in Volatility (squared errors)
- Nonlinear Dependencies
  - After filtering out short-run linear and nonlinear dependencies

The Proposed Model

- The Market Environment
- Description of Economic Agents
  - Determinants of Demands for the Risky Security
  - Hypothesis Development
    - Process
  - Expectation Formation
    - Process

The Market Environment

- Two-asset Market
  - Risk-free bond pays constant interest rate, ***r***
  - Risky stock
  - The stock pays a stochastic dividend of d_t where

$$d_t = \bar{d} + \rho(d_{t-1} - \bar{d}) + \upsilon_t$$

  - Where the error term is distributed $N(0, \sigma_\upsilon^2)$
  - Intuition: Consider this to be a proxy for information flow on a daily basis

Agents

- N agents, each with Exponential (CARA) utility function (constant parameters)
  - Constant absolute risk aversion: λ
- Agents do not differ in terms of utility functions
- Agents have a one-period time horizon and select their demands for the risk asset based upon predictions about the price + dividend and their utility functions
- Agents differ in terms of the hypotheses they hold about the process for the price (+ dividend)
  - An agent's prediction for the next period depends upon how she/he reasons and the hypothesis the agent feels is currently the best for prediction purposes
- Each Agent is endowed with one share: N total shares

Demand for the Security

- Assuming p_{t+1} and d_{t+1} are normally distributed, and single period time horizons, optimal demand by agent *i* for the risky asset equals
- (A) $$x_{i,t} = \frac{E_{i,t}[p_{t+1} + d_{t+1}] - p_t(1+r)}{\lambda \sigma^2_{i,t,p+d}}$$
- Where $E_{i,t}[\bullet]$ should be regarded as the individual agent's prediction based upon the prediction model he selects to use (loosely, a subjective expectation)

Learning and the Formation of Expectations

- Agents are assumed to use a linear forecasting equation:

$$E_{i,t}(p_{t+1} + d_{t+1}) = a_{i,t}(p_t + d_t) + b_{i,t}$$

- Under the assumed dividend process this is the rational forecasting rule in a rational expectations equilibrium
- However, agents in our model cannot be sure that such an equilibrium exists
- We take the structure of the forecasting equation as given
- Agents understand the environment is complex and employ a fuzzy logic learning algorithm to generate model parameter estimates

Price Predictions for any Given Hypothesis

- Agents observe data they then use to form beliefs about the values of the parameters ***a*** and ***b***
- The parameters ***a*** and ***b*** associated with an hypothesis are determined by the ultimate aggregation of the 4 fuzzy rules making up the hypothesis ,
  - e.g. *If {price/fundamental value} is low, then **a** is low and **b** is high*.
  - We shall call the first part the "conditional" part and the second part the "action" part

The Formation of Expectations

- There are **five market descriptors** for the conditional part of the rule:

  [*p*r/d, p/MA(5), p/MA(10), p/MA(100), p/MA(500)*]

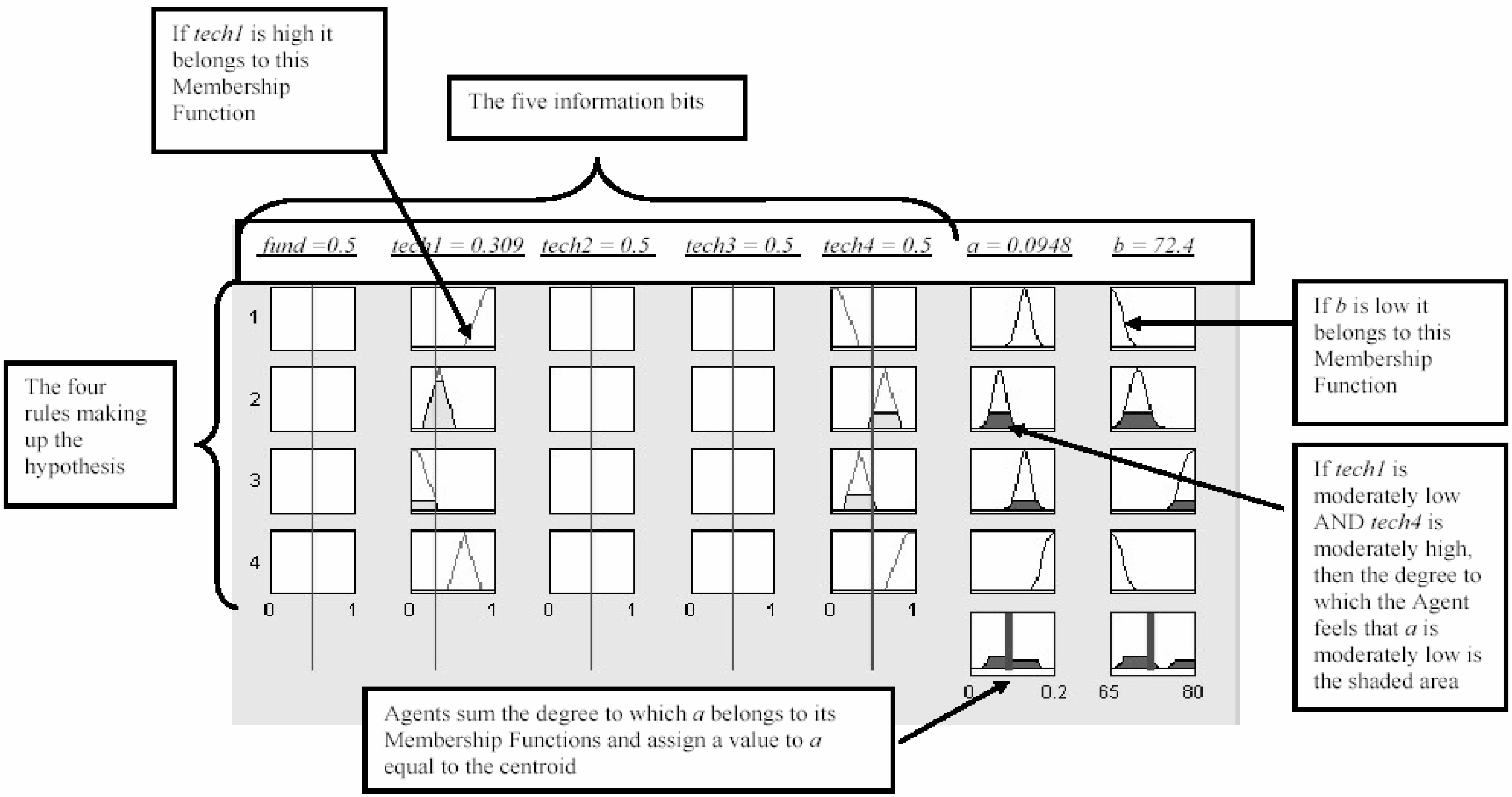
- Each market descriptor is represented by four fuzzy information sets: "*low, moderately low, moderately high, high*" which are coded as "*1, 2, 3, 4*" respectively, a "*0*" is used to record the absence of a descriptor

- Each forecast parameter (***a***, ***b***), is also represented by four fuzzy information sets: "*low, moderately low, moderately high, high*" which are coded as "*1, 2, 3, 4*" respectively

Our Model (Figure 1)

Figure 1

Illustration of the Process By Which an Agent Maps Information Into Conclusions About the Values of the Parameters *a* and *b* Using a Fuzzy Logic Reasoning System



The descriptors, *fund*, *tech2* and *tech3* are excluded from the hypothesis

Rule 1: If *tech1* is high and *tech4* is low, then *a* is moderately high and *b* is low

Rule 2: If *tech1* is moderately low and *tech4* is moderately high, then *a* is moderately low and *b* is moderately low

Rule 3: If *tech1* is low and *tech4* is moderately low, then *a* is moderately high and *b* is high

Rule 4: If *tech1* is moderately high and *tech4* is high, then *a* is high and *b* is low

The Formation of Expectations

- A market hypothesis(a rule base) consists of four fuzzy rules.
- Each agent can entertain up to five different market hypotheses at any given moment.
- In making demand decisions, an agent will utilize the market hypothesis that has recently proven to be the most accurate. Forecast accuracy is given by the inverse of $e^2_{t,i,j}$

$$e^2_{t,i,j} = (1-\theta)e^2_{t-1,i,j} + \theta[(p_t + d_t) - E_{t-1,i,j}(p_t + d_t)]^2$$

Inductive Learning

- Each market hypothesis is assigned a fitness value given by:

$$f_{t,i,j} = -e_{t,i,j}^2 - \beta s$$

  where s is the specificity of the market hypothesis
- Agents revise their market hypotheses on average every τ periods, a GA is used to guide this by subjecting the hypotheses to a process that mimics "reproduction, crossover and mutation" as described earlier where the hypotheses operated on and replaced depend upon their 'fitness'.
- Inductive learning is captured by the constant formulation of new hypotheses, testing of the hypotheses, and the elimination of poor hypotheses.

Simulation Flow

1. Investors form tentative hypotheses about how data on current market condition and past history can be used to forecast next period price and dividend
2. Relying on these hypotheses, investors predict next period price and dividend and determine demands for risky asset
3. Market clearing price is determined and dividend is computed
4. Investors update the accuracies of their hypotheses
5. Steps 2 thru 4 are repeated

Simulation Flow

6. After every τ periods, the hypotheses are revised—the bad hypotheses are discarded and the good hypotheses are combined and modified to produced new hypotheses

The Experiments

- Learning frequency τ = 30 periods
- Combination experiments occur with probability .2
- Individual experiments occur with probability .8
  - Mutation (flipping) of individual elements in the hypothesis equals 0.5
- We simulate the market for 2,500 periods and then record outcomes for the following 2500 periods
- Realized returns are computed in the usual fashion

Conclusion

- The model does a good job reproducing behaviors in security returns that are consistent with observed returns
- We focus on learning, abstracting from market trading structure issues
  - Future research will blend in micro-structure of trading
- The environment is complex and as such agents learn by induction and the application of fuzzy logic, two features we feel are reasonable representations of actual behavior

Essence of the Framework

- Hypothesis development and use based upon a selection process motivated by induction and the individual's reasoning process
- Once the initial conditions are defined, all subsequent events in the virtual financial market are initiated and driven by agent-agent and agent-environment interactions (realized market data)
  - Predictive models (hypotheses) survive because they are perceived as being 'currently' the best
    - Poorly performing hypotheses are discarded or modified
    - New hypotheses are generated and tested
    - Hypotheses are continually checked against the data
  - Agents can hold more than one hypothesis at any time
  - Dynamic, self-referential learning: Agents learn from data they have helped generate, e.g. Market prices

The BDS Test

- Assume we have T observations on a univariate time series $\boldsymbol{x}_t$
- Suppose the series is IID, then the probability of any two consecutive points being less than ε apart in distance will equal a constant

$$\boldsymbol{C_1}(\varepsilon, \boldsymbol{T})$$

- Partition the total time series into 'histories' each of length *n*, which may overlap

The BDS Test

- Consider a comparison of the points in two n-length histories, one beginning at date *t* and one beginning at date $q = t+1$. Define the set of matched points as

$$\left(x_t, x_q\right), \left(x_{t+1}, x_{q+1}\right), \ldots\ldots\ldots\ldots\ldots, \left(x_{t+n-1}, x_{q+n-1}\right)$$

- Define the joint probability of every pair of points satisfying the distance condition by the probability

$$C_n\left(\varepsilon, T\right)$$

The BDS Test

- If the data in the series x_t are actually independent, then for $t \neq q$ the probability that every pair of points is less than ε apart in distance will equal the product of the individual probabilities.
- Moreover, if the x_t are also identically distributed, all of the n probabilities will be the same and each will be the same as the case where n = 1.
- The BDS statistic therefore tests the null hypothesis that

$$C_1(\varepsilon, T)^n = C_n(\varepsilon, T)$$

- which is the null hypothesis that the data are IID

$$BDS_n(\varepsilon, T) = \sqrt{T}\left[C_n(\varepsilon, T) - C_1(\varepsilon, T)^n\right] / \sigma_n(\varepsilon, T) \sim N(0,1)$$

Demands

- Agents in the model know that the demand equation will hold in a homogeneous rational expectations equilibrium when the degree of risk aversion is constant across individuals.
- **However**, the fact that they must use induction to form and modify hypotheses and that they use fuzzy rules (more on this shortly) when forming expectations, means that they never know the true model of prices nor if the market is actually in equilibrium.
- We assume agents select to use (A) when setting their demands, knowing that sometimes the market will be in equilibrium and that sometimes it will not.

Hypothesis Development

- Agents hold multiple hypotheses about prediction models
- They revise these hypotheses over time based upon how well the hypotheses have performed (fitness is a function of squared forecast error and parsimony of the hypothesis)
  - Experiments carried out by agents – formation of new hypotheses
    - Combining elements of existing hypotheses
      - Occurs with probability π =.2
    - Modifying parameters of existing hypotheses
      - Occurs with probability (1-π) =.8

Hypothesis Development

  - Revisions to hypotheses occur ever τ periods
  - An inaccurate hypothesis has a high probability of being replaced by a new hypothesis
- The process is like the mechanism that is found in machine-learning via a genetic algorithm

Structure of a Rule Within an Hypothesis

- A complete rule such as [*0 1 3 0 2 | 2 4*] would mean

  If p/MA(5) is "low", *p/MA(10)* is "moderately high", and *p/MA(500)* is "moderately low" then **a** is "*moderately low*", and **b** is "*high*".

- Notice that since *p*r/d* and *p/MA(100)* are represented by *0*'s, they don't play any role in the conditional part of the rule.