Pricing CDOs with a Generalized non-square Cholesky Decomposition for Selecting Names

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Abstract

The market for credit basket products has become more and more popular and continues to grow. The key for pricing CDOs is to derive the decomposition of default correlation matrix in order to obtain the correlated default time of the entities in the portfolio. The usual way to decompose the correlation matrix is by the algorithm of Cholesky decomposition. However, a Cholesky decomposition cannot work if the given default correlation matrix is a non-positive definite matrix. In this paper, we use a Spectral decomposition that can overcome the shortcoming of a Cholesky decomposition to select the names and then to price the CDO tranches. Our simulation results show that the spreads of CDO trahcnes using a Spectral decomposition are consistent with the spreads by a Cholesky decomposition for the case of positive default correlations among names. In a further step study, we use a Spectral decomposition to select the names of CDOs and to price different trache of CDOs for the case of negative default correlations among names. We find that negative default correlations significantly affect the values of different CDOs tranches. Generally speaking, the spread change due to the correlation change is more sensitive for equity tranches of a high-correlated portfolio. By contrast, the spread change for the super senior tranche is more sensitive in a medium-correlated portfolio.

Keywords: A Cholesky Decomposition, A Spectral decomposition, CDOs

1 Introduction

The credit derivative market has been developed rapidly recently. The British Bankers' Association (BBA) in its Credit Derivative Report 2003/2004 estimated the global credit derivative market (excluding asset swaps) accounted for 3,548 trillion at the end of 2003, increasing to 5,021 billion in 2004 and reach 8,206 billion by 2006. Among the credit derivative product, the top three market shares expected in 2006 are credit default swaps, credit derivative indices and portfolio/synthetic CDOs (Collateralized Debt Obligations).

CDOs are asset-backed securities where the underlying portfolio including either various types of debt obligations or focus solely on one class of debt such as bonds or loans. For synthetic CDOs, the collateral pool is a collection of single-name credit default swaps. Based on portfolio credit quality, for example, the debt tranches are usually classified into senior tranche, mezzanine tranche and equity. The loss triggered by the credit events, such as default, is distributed by the bottom-up sequence through the tranches. That is, as long as any loss occurs in the profile of collateral pool, the equity first absorbs the losses up to some extent.

For the credit basket products, the important issue in the pricing procedures is to determine the default correlation parameters appearing in the model. Hull and White (2001) extend the structure model to construct a multi-name pricing model and provide a methodology which takes counterparty default risk into account for pricing credit basket products. For the reduced form models, Duffie and Gârleanu (2001) address the dependence of default times through correlated stochastic risk intensities to pricing the CDO tranches.

Li (1999, 2000) uses a normal copula function to combine the marginal default probability of an underlying asset into the correlated distribution for all the assets in a basket. Meneguzzo and Vecchiato (2004) test the copula sensitivity of credit derivatives by assuming some alternative specifications of the joint distributions of default risks. Other related reference, such as Bouyé et al (2000), Durrleman et al (2000), Bluhm (2003), Hull and White (2004), Friend and Rogge (2004), Giesecke (2004), and Cherubini et al (2004), also consider the applications of copulas in the pricing of basket credit derivatives.

Nowadays CDOs are well known in the financial markets. Many practitioners use copula functions for pricing and risk monitoring CDOs. A normal copula function is one of the first approach and represents the current market standard in modeling portfolio credit risks. Li (1999, 2000) employed the normal copula function to combine the individual default probability for each name into the correlated distributions for all of the companies in a basket. One can then simulate the correlated default time of the entities from the joint distribution and determine the price of credit basket products.

The default correlation matrix is decomposed in the simulation algorithm processes in order to obtain the correlated default time of the entities. That is, we need to find a matrix A such that $AA^T=R$, in which R represents the correlation matrix and A^T is the transposition of matrix A. The common way to derive matrix A is by a Cholesky decomposition. However, a Cholesky decomposition cannot work if the given correlation matrix is a non-positive definite matrix (Rebonato & Jackel (1999) and Reiβ (2002)). To solve the above mentioned problem, Rebonato and Jackel (1999) propose a Spectral decomposition which can decompose the correlation matrix with both of the negative or positive definite matrix.

The contributions of our paper are twofold. First, we show that a Spectral decomposition performs as well as a Cholesky decomposition does when the default correction matrix is a positive define matrix. We then demonstrate that a Spectral decomposition can still work when the default correction matrix is a non-positive define matrix by simulations. Second, we then employ a Spectral decomposition to choose the names of and CDOs and to price these products. Our simulation results show that negative default correlation significantly affects the values of different CDOs tranches. Generally speaking, spread changed due to correlation change is more sensitive in high-correlative portfolio for equity tranch. By contrast, the spread changed for super senior tranche is more sensitive in medium-correlative portfolio. To our best knowledge, our paper is the first one to price CDOs when the names have non-positive definite correlation matrix appeared often in the real markets.

The rest of this paper is organized as follows. Section 2 describes the valuation models, including literature models and our model, for CDO tranches. Section 3 presents and analyzes the simulation results using numerical examples. Section 4 concludes the paper.

2 The Valuation of CDO

2.1 Copula Functions

A copula function is a mathematical function that combines univariate marginal distribution into the full multivariate distribution. A copula of n variables is a function C defined on $[0,1]^n$ with range on [0,1]. Let $f_i(y_i)$, $F_i(y_i)$ be the univariate marginal distribution function and the cumulative distribution function for each elements Y_i at point y_i respectively, for $i=1,2,\dots,n$. The multivariate distribution F with given marginal distributions, $F_1, F_2,\dots,F_n$ can be determined by using copula function C, or

$$F(y_1, y_2,\dots, y_n) = \mathrm{C}(F_1\ (y_1)\ ,\ F_2\ (y_2)\ \dots,\ F_n\ (y_n)). \qquad (1)$$

Some common copula functions are normal copula, t copula, Frank Copula.

The normal copula function is defined as:

$$C(u_1,u_2,...,u_n) = \Phi_n(\Phi^{-1}(u_1),\Phi^{-1}(u_2),...,\Phi^{-1}(u_n);\rho)\,, \qquad (2)$$

where u_i is the univarite cumulative probability at $Y_i = y_i$, Φ_n is the n multivariate normal distribution function with correlation coefficient ρ, and Φ^{-1} is the inverse of a univariate normal distribution function. The student's t copula function is defined as:

$$C(u_1,u_2,...,u_n) = T_v(T_v^{-1}(u_1),T_v^{-1}(u_2),...,T_v^{-1}(u_n);\rho)\,, \qquad (3)$$

where T_v is the multivariate student's t distribution function with v degree of freedom, and T_v^{-1} is the inverse of the univariate student's t distribution with v degree of freedom.

2.2 The Li (2000) Model

In pricing a CDO, it is required the model of joint default losses in a portfolio. Li (2000) provided the copula approach to solve the modeling problem of a joint loss distribution. A random variable, *T,* called time-until-default measures the period from today to the default. The survival function *S(t)* represents the probability that an firm will survive until time *t*. The survival function can be expressed in terms of a hazard rate function. Let *F(t)* be the distribution function of *T*:

$$F(t) = \Pr(T \le t) \qquad t \ge 0 \qquad (4)$$

and

$$S(t) = \Pr(T > t) = e^{-\int_0^t h(u)du}\,, \qquad (5)$$

where

$$h(t) = \Pr(t < T \le t + \Delta t \mid T > t)\,.$$

The hazard function *h(t)* represents the instantaneous default probability of an issuer that has survived until time t. Based on the common assumption of a constant hazard rate, the survival time follows an exponential distribution with parameter h, or

$$S(t) = e^{-ht}, \tag{6}$$

and

$$f(t) = he^{-ht}. \tag{7}$$

The hazard rate can be estimated by extracting the default probabilities using observable market data including asset swap spreads or the prices of risky corporate bonds (Li (1998)). After the parameter of h is estimated, the distribution of the survival time of each issuer can be specified.

The distribution of a credit portfolio determines the value of the tranche of the synthetic CDO. Li (2000) use a normal copula to link the marginal default probabilities of an individual name to the joint distribution of the survival times, as follows.

$$F(t_1, t_2, \ldots, t_n) = \Phi_n(\Phi^{-1}(F_1(t_1)), \Phi^{-1}(F_2(t_2)), \ldots, \Phi^{-1}(F_n(t_n))), \tag{8}$$

The main task is to simulate correlated survival times T. Let

$$Y_i = \Phi^{-1}(F_i(t_i)). \tag{9}$$

Y_i is simulated instead of T_i because there is one-to-one mapping between Y and T. Based on the Li (2000) model, the simulation algorithm is summarized as follows:

1. Find the Cholesky decomposition A of correlation matrix R.
2. Simulate n independent random variables $Z = (Z_1, Z_2, \ldots, Z_n)'$ from normal distribution.
3. Let $Y=AZ$, then Y follows multivariate normal with correlation R.
4. Obtain $(T_1, T_2, \ldots, T_n)$ by mapping back the Y's to the T's using

$$T_i = F^{-1}(\Phi(Y_i)) = -\frac{\ln(1-\Phi(Y_i))}{h_i}, \tag{10}$$

where Φ is the normal cumulative distribution function; h_i is the hazard rate function of i issuers and can be estimated by the equation

$$h = \frac{CDS\ \text{Spread}}{1\ \text{- Recovery Rate}}. \tag{11}$$

Each simulation run yields correlated survival times for all credits in the portfolio. The sample path of the time-until-default takes into account two factors - the credit quality of an issuer (related to the hazard rate h) and the correlation matrix R (which influences the procedure for sampling the random variables Y_i). This simulated default time of each issuer is used to determine the portfolio loss distribution, and then to price the tranche of CDO.

2.3 The Valuation of synthetic CDOs

The fair premium of particular tranche is set to equate the value of the default leg to the value of premium leg. Given the survival times $(t_1, t_2, \ldots, t_n)$ for all credits on a simulation run, the default loss on each tranche can be obtained. The loss rate for a particular tranche at time t ($LR_{tranche}(t)$) is

$$LR_{tranche}(t) = \frac{\min(\max(LR(t) - P_{low}, 0), P_{high} - P_{low})}{P_{high} - P_{low}}. \tag{12}$$

where P_{low} and P_{high} are the proportions of the default losses born by this specific tranche, $LR(t)$ is the cumulative loss rate of an N-name CDO at time t and

$$LR(t) = \sum_{i=1}^{N} 1_{\{t_i < t\}}(1 - R_i) / N, \tag{13}$$

where $1_{\{t_i < t\}}$ is an indicator function to denote the counting process which jumps from 0 to 1 at default time of name i. The expected loss rate of the given tranche at present time can be written as

$$E^{P}\left[\int_0^T B(0,t)dLR_{Tranche}(t)\right], \tag{14}$$

where P denotes the risk-neutral probability measure; $B(0,t)$ denotes the discount factor for the maturity t; and T is the maturity of the CDO. For the premium leg, let $\{m_0, m_1, m_2, \ldots, m_n\}$ be the series of premium payment dates. Clearly, $m = 0$ and $m_n = T$. The value of premium leg at present time (ignore the accrued premium payments between payments dates) is written as

$$E^{P}\left[\sum_{i=1}^{n}(m_i - m_{i-1})WB(0,m_i)\min\{\max[P_{high} - LR_{Tranche}(m_i),0], P_{high} - P_{low}\}/(P_{high} - P_{low})\right] \tag{15}$$

where W is the fair spread. Set Eq. (14) equate to Eq. (15), the fair spread W can be determined.

2.4 A Spectral Decomposition for Valuing CDOs

As mentioned earlier, a Cholesky decomposition adopted by Li can not work when the default correlation matrix is not a positive-definite matrix. Recently, Rebonato & Jackel (1999) proposed a Spectral decomposition method that does not require a pre-existing positive-definite matrix to start with. Define S as eigensystem, $\{\lambda_i\}$ as associated set of eigenvalues of the correlation matrix C such that

$$C \cdot S = \Lambda \cdot S \quad \text{with} \quad \Lambda = diag(\lambda_i). \tag{16}$$

Define the non-zero elements of the diagonal matrix Λ^* as

$$\Lambda^* : \lambda_i^* = \begin{cases} \lambda_i & \text{if } \lambda_i \geq 0 \\ 0 & \text{if } \lambda_i < 0 \end{cases}. \tag{17}$$

If the target matrix C is not positive-semidefinite, it has at least one negative eigenvalue and at least one of the λ_i^* will be zero. Define the non-zero elements of the diagonal scaling matrix T with respect to the eigensystem S as

$$T : t_i = \left[\sum_m s_{im}^2 \lambda_m^* \right]^{-1}, \tag{18}$$

and

$$B^* = S\sqrt{\Lambda^*} . \tag{19}$$

Let

$$B = \sqrt{T}\, B^* = \sqrt{T}\, S \sqrt{\Lambda^*} , \tag{20}$$

and construct the matrix $\tilde{C}$ by

$$\tilde{C} = BB^T . \tag{21}$$

$\tilde{C}$ is the unique best semi-positive approximate of C with respect to the Frobenius norm if for any positive symmetric matrix $\overset{*}{C}$ with $\overset{*}{C} \neq \tilde{C}$, the following inequation is hold.

$$\left\| C - \tilde{C} \right\|_F < \left\| C - \overset{*}{C} \right\|_F \tag{22}$$

$\|A\|_F$ is the Frobenius norm of an $m \times n$ matrix A which is the matrix norm defined as the square root of the sum of the absolute squares of its elements,

$$\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} \left| a_{ij} \right|^2} . \tag{23}$$

Higham (1988, theorem 2.1) proved that Eq(22) always exists. We briefly restate the proof in the Appendix.

If the approximated matrix $\tilde{C}$ is not the same as the target matrix C, there would exist some errors. The error measure ε is defined as the difference of two matrices $\tilde{C}$ and C, or

$$\varepsilon_{ij} = \left(C_{ij} - \tilde{C}_{ij} \right). \tag{24}$$

Define χ as the squared sum of the elements of error matrix ε,

$$\chi = \sum_{ij} \varepsilon_{ij}^{2} \quad . \tag{25}$$

In the following context, we use an example to demonstrate the procedures described above. We define a target matrix C as follows:

$$C = \begin{pmatrix} 1 & 0.9 & 0.7 \\ 0.9 & 1 & c_{23} \\ 0.7 & c_{32} & 1 \end{pmatrix}. \tag{26}$$

c_{23} is equal to c_{32} because C is a symmetric matrix. The values of c_{23} vary from 0.4 to 0 in order to examine the relationship between the eigenvalue and the error measure. Table 1 shows the target matrix, the associated eigenvalues, the decomposed matrix B obtained by a Spectral decomposition, the approximated matrix and the error measure χ under the different value of c_{23}.

[Insert Table 1 Here]

C is a positive-definite, or all of the eigenvalues are greater than zero, when c_{23} is set to 0.4 and the error measure χ is zero which means the approximate matrix is perfect match to the target matrix. The error measure χ is bigger than zero when there at least one eigenvalue is negative. The smallest eigenvalue of target matrix and the error measure χ under different value of c_{23} are shown in Figure 1.

[Insert Figure 1 Here]

Figure 1 shows a negative relation between eigenvalues and the error measures χ. That is, the smaller negative eigenvalue the target matrix has, the more the error measure is, and the more dissimilar the approximate matrix to the target matrix. When all of the eigenvalues of the target matrix are non-negative, the approximated matrix will be identical to the target matrix.

We have showed that the Spectral decomposition can work even the target matrix is not a positive-semidefinite matrix. Now, we further examine the Spectral decomposition in another way. A matrix can be both decomposed by Cholesky decomposition and Spectral decomposition. Is it producing different results of those two methods in Monte Carlo simulation? If the difference is not toot much, we may use a Spectral decomposition instead of a Cholesky decomposition to decompose matrix in all situations.

We choice twenty funds as shown in Table 2 and collect their weekly closed prices in the sample period form 1 Jan, 2003 to 31 Dec. 2003. The correlation matrix is estimated by the historical time series data. This 20 by 20 correlation matrix just can be both decomposed by Cholesky decomposition and Spectral decomposition that will help us to examine the different simulated results of those two methods.

[Insert Table 2 Here]

Follow Li (2000) simulation algorithm, we simulate the correlated survival times of 20 funds by two decomposition methods. Let Dc_i and Ds_i be the simulated survival time for i^{th} fund by Cholesky decomposition and Spectral decomposition respectively. Define error measure as

$$\sum_{i=1}^{20} error_i = \sum_{i=1}^{20} \left(Dc_i - Ds_i\right)^2 . \quad (27)$$

Two methods have more different when the errors are large. We run the simulations with different numbers: 100, 200, 500, 1000 and 5,000. Figure 2 shows that error of every fund as adding simulation numbers to run.

[Insert Figure 2 Here]

The square sums of the difference results obtained by two methods are calculated. Figure 3 shows the relation between the errors and simulation runs. From Figure 3, the deviations of Dc_i from Ds_i are decreasing as the number of simulation increasing. When the simulation number reaches to 5,000, the deviation is very slight. That is, the simulation results are almost the same when the number of simulation is large enough. Based on this result, we may use Spectral decomposition in the pricing of basket credit products if we keep the simulation run large. Following we set the number of simulation to 30,000 in the pricing of CDO.

[Insert Figure 3 Here]

3 Simulation Results

We investigate how the default correlation changes affect the values of CDO tranches in this section. Suppose that the portfolio comprising 50 default swaps each with one million notional amount. The pool of the default swaps is assumed to be homogeneous in terms of spread, recovery and correlation assumptions. This portfolio is partitioned into a structure of four tranches: equity, mezzanine, senior and super senior. The attachment points, at which the underlying portfolio begins to absorb the loss due to default, of the tranches are 0%, 5%, 10% and 15% for equity, mezzanine, senior and super senior tranches, respectively. Table 3 decsribes the underlying collateral pool and tranches in details.

[Table 3 Here]

The loss distribution is bottom-up sequentially through the tranches. For example, equity tranche absorbs the first 5% of losses, or 2.5 million of notional amounts, on the collateral pool if defaults happen. The mezzanine tranche absorbs losses from 5% to 10% on the portfolio due to default, and so on.

The default correlation matrix is decomposed using two decomposition methods, a Cholesky decomposition and a Spectral decomposition, to examine the difference of tranche valuation. If the values obtained from these two methods are not much different, we may use a Spectral decomposition instead of a Cholesky decomposition to price the CDO tranches for the merit that the former can decompose the default correlation without the requirement of a positive-semi definition matrix.

We use the normal copula, which is the market standard in modeling portfolio credit risk, to combine the individual default probability into the correlated distributions for all of the companies. The results are listed in Table 4. The credit risk of the underlying portfolio depends on the attachment point of the tranche. The premium of the equity tranche is higher than that on any other tranche at all levels of correlation because it is associated with no subordination. By contrast, the premium of the super senior tranche is the lowest because the super senior tranche has the largest loss protection.

[Insert Table 4 Here]

The spread of equity and super senior tranches is a monotonic function with respect to default correlation. The premium on the equity tranche decreases as the default correlation increases. However, the premium on the super senior tranche increases with the correlation. The pattern of the premium for the mezzanine and senior tranches lies between these two tranches. The spreads of mezzanine and senior tranches increase first and then decrease when the default correlation increases.

As shown in Table 4, the spread of mezzanine tranche increases when the default correlation is less than 0.2, whereas it decreases when the default correlation is more then 0.2. For the senior tranche, the premium is the largest when the default correlation is at 0.6 which is higher than the default correlation of mezzanine tranche because the senior tranche has more subordination then mezzanine tranche does.

The spread differences are defined as the differences between the fair spreads obtained by a Cholesky decomposition and these by a Spectral decomposition as shown in the 10th through the 13th columns in Table 4 for equity, mezzanine, senior and super senior tranches, respectively. The spread differences are less than 10 basis points for all the level of default correlations and for all tranches. Therefore, the pricing results based on those two decompositions are not significantly different.

We also test the deviations of fair spread obtained by a Spectral decomposition from the fair spread by a Cholesky decomposition using a *RMSE* measure which is defined as

$$RMSE(in\,\%) = \sqrt{\frac{\sum_{i=1}^{n}\left(S_{S,i} - S_{C,i}\right)^2}{n}} \times 100\%, \tag{28}$$

where $S_{S,i}$ and $S_{C,i}$ denote the fair spread for i^{th} simulation by a Spectral decomposition and a Cholesky decomposition, respectively, and n is the total simulation number, in our example n is set to 50. As shown in Table 4, the value of *RMSE* falls in the range from 0.1601% to 0.0012% in four tranches and in different correlation assumptions.

Furthermore, we examine the default correlation effects, including negative correlations, on the pricing of CDO tranches. The default correlation is homogeneous when the values are one in the diagonal and r off the diagonal of the correlation matrix. The negative correlation effect is examined by changing one of the *r's* in the matrix. We assume the coefficient of correlation between name *1* and name *2* is r_{12} instead of r. The default correlation is a 50 by 50 symmetrical matrix as follows:

$$\begin{bmatrix} 1 & r_{12} & r & \cdots\cdots & r \\ r_{12} & 1 & r & \cdots\cdots & r \\ r & r & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots \quad \ddots & \vdots \\ \vdots & \vdots & & \ddots \quad \ddots & r \\ r & r & \cdots\cdots & r & 1 \end{bmatrix} . \tag{29}$$

The negative correlation effects are examined by some possible numbers of r_{12}: -0.9, -0.7, -0.5, -0.3, and -0.1. The portfolio is assumed high-correlative assets in order to avoid the portfolio diversification diluting the negative correlation effects. Thus, the values of r are chosen from 0.5 to 0.9. The fair spreads of equity and super senior tranches under different assumptions of default correlation are shown in Table 5.

[Insert Table 5 Here]

The results considering negative correlations are consistent with those of Table 4, which only presents positive correlation effects. For an equity tranche, the fair spread is negatively correlated with the default correlations in a high-correlated portfolio. From Table 5, it is interesting to note that when the r_{12} is at value of -0.9 (close to perfect negative correlation), the spread of equity is the largest and the spread of super senior tranche is the smallest for all of the r changed.

Percent change measures the spread change in terms of percentage due to r_{12} changes under the same portfolio correlation assumption. For example, percent change for r = 0.5 and r_{12} = -0.7 is calculated by:

$$\%\ change_{r=0.5, r_{12}=-0.7} = \frac{S_{r=0.5, r_{12}=-0.7} - S_{r=0.5, r_{12}=-0.9}}{S_{r=0.5, r_{12}=-0.9}} , \tag{30}$$

where S denotes the fair spread. For an equity tranche, the average % change in the high-correlated portfolio is higher than that in a medium-correlated portfolio. For example, the average % change is -0.6294% when r = 0.9, and -0.3416% when r = 0.5. The result is in a reversed direction for a super senior tranch. The spread changed in a medium-correlated portfolio is more sensitive than in a high-correlated portfolio. The average % change is 0.4123% when r = 0.5, and 0.2046% when r = 0.9.

We summarize our simulation results as follow. We find that negative default correlations significantly affect the values of different CDOs tranches. Generally speaking, the spread change due to the correlation change is more sensitive for equity tranches of a high-correlated portfolio. By contrast, the spread change for the super senior tranche is more sensitive in a medium-correlated portfolio.

4 Conclusions

In this paper, we first demonstrate that a Spectral Decomposition method can decompose a positive definite correlation matrix as a Cholesky decomposition does, yet a Spectral Decomposition method can overcome the shortcoming of a Cholesky decomposition. We then use a Spectral decomposition method to choose the names and to price CDOs when the default correlation matrix is a non-positive definite matrix. We also investigate how the changes on the correlation coefficients among names affect the values of different tranches in a CDO structure.

In general, we find that negative default correlations significantly affect the values of different CDOs tranches. Generally speaking, the spread change due to the correlation change is more sensitive for equity tranches of a high-correlated portfolio. By contrast, the spread change for the super senior tranche is more sensitive in a medium-correlated portfolio. To our best knowledge, our study is the first one to investigate how to choose the names of CDOs and to price different tranche when the default correlation matrix is a non positive definite matrix. Hence our study provides a more general method for practitioners and academic researchers to value CDOs.

Appendix：The Frobeninus Norm Positive Approximant

The Frobenius norm is defined to measure the distance between every vector that is:

$$\|A\|_F = (\sum_{i,j} a_{ij}^2)^{1/2} \text{, with aij is vector of A.} \qquad \text{(A.1)}$$

Recall that a symmetric matrix A is positive definite if its eigenvalues are positive, and positive semidefinite, which we will denote by *A*≧*0*, if its eigenvalues are non-negative. The distance in the norm ||.|| form an arbitrary *A* to the set of positive definite matrices is denoted by

$$\delta(A) = \min_{X = X^T \geq 0} \|A - X\|, \qquad \text{(A.2)}$$

and any positive definite *X* satisfying $||A\text{-}X||=\delta(A)$ is termed a positive approximant of *A* in given norm.

Let $A \in R^{n \times n}$, and let $B = (A + A^T)/2$ and $C = (A - A^T)/2$ be the symmetrix and skew-symmetrix parts of *A* respectively. Let $B = UH$ be a polar decomposition ($U^T U = I, H = H^T \geq 0$). Then *XF= (B+H)/2* is the unique positive approximant of *A* in the Frobenius norm, and

$$\delta_F(A)^2 = \sum_{\lambda_i(B)<0} \lambda_i(B)^2 + \|C\|_F^2. \qquad \text{(A.3)}$$

Proof **:**

Let *X* be positive definite. Form the fact that $\|S + K\|_F^2 = \|S\|_F^2 + \|K\|_F^2$ if $S = S^T$ and $K = -K^T$, we have

$$\|A - X\|_F^2 = \|B - X\|_F^2 + \|C\|_F^2, \qquad \text{(A.4)}$$

and so the problem reduces to that of approximating *B*. Let $B = Z\Lambda Z^T$ ($Z^T Z = I$, $\Lambda = \text{diag}(\lambda_i)$) be a Spectral decomposition, and let $Y = Z^T XZ$. Then

$$\|B - X\|_F^2 = \|\Lambda - Y\|_F^2$$

$$= \sum_{i \neq j} y_{ij}^2 + \sum_i (\lambda_i - y_{ii})^2$$

$$\geq \sum_{\lambda_i < 0} (\lambda_i - y_{ii})^2 \geq \sum_{\lambda_i < 0} \lambda_i^2 \qquad \text{(A.5)}$$

since $y_{ii} \geqq 0$ because Y is positive definite. This lower bound is attained, uniquely, for the matrix $Y=diag(d_i)$, where

$$d_i = \begin{cases} \lambda_i, \lambda_i \geq 0, \\ 0, \lambda_i < 0, \end{cases} \qquad \text{(A.6)}$$

that is,

$$X_F = Zdiag(d_i)Z^T . \qquad \text{(A.7)}$$

The representation $X_F=(B+H)/2$ follows, since $H = Zdiag(|\lambda_i|)Z^T$.

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Table 1：The Decomposition Error of a Matrix using a Spectral Decomposition

C is the target matrix, B is the decomposed matrix obtained by a Spectral decomposition, $\tilde{C} = BB^T$ is the approximate matrix. χ is the error measure and define as

$$\chi = \sum_{ij} \varepsilon_{ij}^2 \quad \varepsilon_{ij} = \left(C_{ij} - \tilde{C}_{ij} \right)$$

C			Eigenvalue	B			$\tilde{C} = BB^T$			χ
1	0.9	0.7	0.030347	0.13191	0.087183	0.98742	1	0.9	0.7	0
0.9	1	0.4	0.61602	-0.10021	0.45536	0.88465	0.9	1	0.4	
0.7	0.4	1	2.3536	-0.05389	-0.63329	0.77203	0.7	0.4	1	
1	0.9	0.7	-0.00735	0	0.062382	0.99805	1	0.89402	0.69632	0.000100486
0.9	1	0.3	0.71062	0	0.50292	0.86433	0.89402	1	0.30097	
0.7	0.3	1	2.2967	0	-0.6729	0.73974	0.69632	0.30097	1	
1	0.9	0.7	-0.01133	0	0.060076	0.99819	1	0.89083	0.6943	0.00023784
0.9	1	0.29	0.72017	0	0.50703	0.86193	0.89083	1	0.29153	
0.7	0.29	1	2.2912	0	-0.67667	0.73629	0.6943	0.29153	1	
1	0.9	0.7	-0.01535	0	0.057803	0.99833	1	0.88764	0.69226	0.00043443
0.9	1	0.28	0.72973	0	0.51108	0.85954	0.88764	1	0.28213	
0.7	0.28	1	2.2856	0	-0.68043	0.73282	0.69226	0.28213	1	
1	0.9	0.7	-0.0194	0	0.055563	0.99846	1	0.88445	0.69019	0.000691307
0.9	1	0.27	0.7393	0	0.51506	0.85716	0.88445	1	0.27276	
0.7	0.27	1	2.2801	0	-0.68417	0.72933	0.69019	0.27276	1	
1	0.9	0.7	-0.02348	0	-0.05335	0.99858	1	0.88126	0.68808	0.00101009
0.9	1	0.26	0.74888	0	-0.51897	0.85479	0.88126	1	0.26343	
0.7	0.26	1	2.2746	0	0.68789	0.72582	0.68808	0.26343	1	
1	0.9	0.7	-0.0276	0	-0.05117	0.99869	1	0.87808	0.68596	0.001389334
0.9	1	0.25	0.75847	0	-0.52282	0.85244	0.87808	1	0.25413	
0.7	0.25	1	2.2691	0	0.69159	0.72229	0.68596	0.25413	1	
1	0.9	0.7	-0.04866	0	-0.04057	0.99918	1	0.86217	0.67496	0.004248068
0.9	1	0.2	0.80654	0	-0.54118	0.84091	0.86217	1	0.20812	
0.7	0.2	1	2.2421	0	0.70986	0.70434	0.67496	0.20812	1	
1	0.9	0.7	-0.09304	0	-0.02033	0.99979	1	0.83056	0.65155	0.015035
0.9	1	0.1	0.90312	0	-0.5737	0.81907	0.83056	1	0.11866	
0.7	0.1	1	2.1899	0	0.74521	0.66683	0.65155	0.11866	1	
1	0.9	0.7	-0.14018	0	-1.21E-16	1	1	0.79928	0.62686	0.033143
0.9	1	0	1	0	-0.60096	0.79928	0.79928	1	0.032817	
0.7	0	1	2.1402	0	0.77913	0.62686	0.62686	0.032817	1	

Table 2: The Name List of the Fund

No.	Name of the Fund	No.	Name of the Fund
1	DIT Wachstum Global	11	INVESCO GT European Bond A
2	DIT Wachstum Europa	12	Schroder ISF Euro Eq A Inc
3	DIT Wachstum Plus	13	Schroder ISF US Sm Cos A Inc
4	Templeton Gbl Bond A dis	14	Schroder ISF Gl Eq Sig A Inc
5	DIT Finanzwerte	15	Schroder ISF Eur Eq Sig A Inc
6	DIT Softwarefonds	16	DIT Multimedia
7	Baring Hgh Yld Bd Hdgd	17	Franklin Capital Gth/A
8	DIT Biotechnologie	18	INVESCO GT Cont Europe A
9	INVESCO GT Asia Enterprise A	19	INVESCO Maximum Income A
10	Schroder ISF North Amer A Inc	20	INVESCO GT Amer Enterprise A

Table 3: The CDO Portfolio and Tranches

Collateral Description		Tranching	
Number of Obligors	50	Equity	0%-5%
Notional Per Credit	1m	Mezzanine	5%-10%
Total Portfolio Size	50m	Senior	10%-15%
Maturity (years)	5	Super Senior	15%-100%
CDS Spread on each credit (bps)	100		
Recovery Rate on each credit	35%		

Table 4: The Spread of each Tranche in a CDO structure with Positive Default Correlation

The hypothetical portfolio with an underlying collateral pool and tranches is shown in Table 3. The number of simulation is 50 and there are 30,000 runs in each simulation. Standard errors are in parenthesis below the fair spread (in basis points). The interest rate is assumed a constant rate of 2%.

Default Correlation	Cholesky Decomposition				Spectral Decomposition				Spread Differences (in bps) RMSE (in %)			
	Equity	Mezzanine	Senior	Super	Equity	Mezzanine	Senior	Super	Equity	Mezzanine	Senior	Super
0.1	3099.23	514.14	102.06	1.35	3106.72	519.54	104.27	1.47	-7.49	-5.39	-2.21	-0.12
	(18.61)	(3.57)	(0.40)	(0.07)	(12.75)	(2.93)	(0.77)	(0.04)	0.1048%	0.0637%	0.0249%	0.0012%
0.2	2387.23	545.21	177.79	5.53	2378.16	552.99	183.00	5.73	9.07	-7.78	-5.21	-0.20
	(11.37)	(3.28)	(2.09)	(0.17)	(7.37)	(3.87)	(1.81)	(0.08)	0.1090%	0.0826%	0.0539%	0.0022%
0.3	1908.77	541.18	226.78	11.37	1904.28	546.39	232.20	11.74	4.49	-5.22	-5.42	-0.37
	(11.06)	(2.53)	(1.66)	(0.37)	(7.73)	(3.14)	(2.26)	(0.20)	0.1601%	0.0628%	0.0553%	0.0043%
0.4	1547.79	522.85	257.31	18.03	1549.88	526.59	262.94	18.66	-2.09	-3.74	-5.63	-0.63
	(7.55)	(1.20)	(1.39)	(0.47)	(5.11)	(1.87)	(2.68)	(0.15)	0.0584%	0.0386%	0.0581%	0.0071%
0.5	1263.23	494.37	274.51	25.40	1258.59	496.14	279.03	26.38	4.64	-1.77	-4.52	-0.97
	(4.08)	(0.70)	(0.30)	(0.61)	(3.41)	(3.09)	(2.76)	(0.27)	0.0691%	0.0405%	0.0523%	0.0103%
0.6	1022.29	457.62	280.22	33.40	1016.41	456.32	285.13	34.49	5.88	1.29	-4.91	-1.09
	(3.37)	(2.57)	(1.51)	(0.72)	(4.65)	(2.88)	(2.39)	(0.11)	0.0650%	0.0551%	0.0528%	0.0126%
0.7	812.27	412.91	277.77	42.32	806.53	412.73	282.66	43.76	5.74	0.18	-4.89	-1.44
	(3.96)	(3.07)	(1.66)	(0.72)	(2.29)	(3.06)	(1.79)	(0.13)	0.0622%	0.0550%	0.0514%	0.0156%
0.8	621.03	359.56	264.79	52.49	617.61	361.84	269.72	54.02	3.43	-2.28	-4.93	-1.53
	(4.62)	(2.97)	(2.84)	(0.76)	(2.78)	(0.87)	(1.80)	(0.05)	0.0668%	0.0431%	0.0623%	0.0169%
0.9	431.77	295.01	239.06	64.87	430.53	298.60	242.70	66.51	1.24	-3.59	-3.65	-1.64
	(3.86)	(4.90)	(3.76)	(0.69)	(4.11)	(1.67)	(1.62)	(0.04)	0.0728%	0.0643%	0.0534%	0.0177%

Table 5: The Spread of each tranche in a CDO with Negative Default Correlations

The hypothetical portfolio with an underlying collateral pool and tranches is shown in Table 3. The number of simulation is 50 and there are 30,000 runs in each simulation. Standard errors are in parenthesis below the fair spread (in basis points). The interest rate is assumed a constant rate of 2%. S denotes for fair spread. % change measures correlation sensitivity on the spread. For example, % change($r = 0.5, r_{12} = -0.7$) $= (S_{r=0.5,r12=-0.7} - S_{r=0.5,r12=-0.9}) / S_{r=0.5,r12=-0.9}$.

r_{12}	r = 0.5 S	0.5 % change	0.6 S	0.6 % change	0.7 S	0.7 % change	0.8 S	0.8 % change	0.9 S	0.9 % change
Equity										
-0.9	1279.68	_-	1037.98	-	832.91	-	645.23	-	465.19	-
	(9.49)		(10.21)		(8.53)		(8.32)		(5.31)	
-0.7	1274.44	-0.4089%	1036.81	-0.1129%	830.12	-0.3345%	642.91	-0.3608%	462.77	-0.5194%
	(8.35)		(9.07)		(8.90)		(7.92)		(5.02)	
-0.5	1271.66	-0.2187%	1033.81	-0.2890%	826.55	-0.4302%	639.66	-0.5057%	459.77	-0.6493%
	(8.79)		(10.18)		(9.16)		(7.79)		(5.20)	
-0.3	1264.56	-0.5583%	1029.17	-0.4492%	820.43	-0.7402%	636.35	-0.5160%	456.95	-0.6129%
	(8.71)		(10.05)		(9.17)		(7.75)		(5.03)	
-0.1	1262.27	-0.1807%	1025.04	-0.4013%	819.25	-0.1439%	632.29	-0.6388%	453.59	-0.7359%
	(8.56)		(9.76)		(8.56)		(7.82)		(5.28)	
Average		-0.3416%		-0.3131%		-0.4122%		-0.5053%		-0.6294%

r_{12}	r = 0.5 S	0.5 % change	0.6 S	0.6 % change	0.7 S	0.7 % change	0.8 S	0.8 % change	0.9 S	0.9 % change
Super Senior Tranche										
-0.9	25.40	-	33.20	-	42.21	-	52.05	-	63.78	-
	(0.65)		(0.76)		(0.85)		(1.05)		(1.29)	
-0.7	25.49	0.3456%	33.55	1.0768%	42.30	0.2114%	52.11	0.1147%	63.88	0.1570%
	(0.62)		(0.69)		(0.84)		(1.05)		(1.30)	
-0.5	25.59	0.3925%	33.68	0.3692%	42.43	0.3084%	52.22	0.2021%	64.01	0.2052%
	(0.62)		(0.69)		(0.84)		(1.06)		(1.29)	
-0.3	25.60	0.0542%	33.78	0.3166%	42.14	-0.6975%	52.36	0.2657%	64.16	0.2228%
	(0.64)		(0.69)		(0.90)		(1.06)		(1.28)	
-0.1	25.82	0.8569%	33.91	0.3809%	42.67	1.2746%	52.50	0.2832%	64.31	0.2334%
	(0.61)		(0.70)		(0.86)		(1.09)		(1.30)	
Average		0.4123%		0.5359%		0.2742%		0.2164%		0.2046%

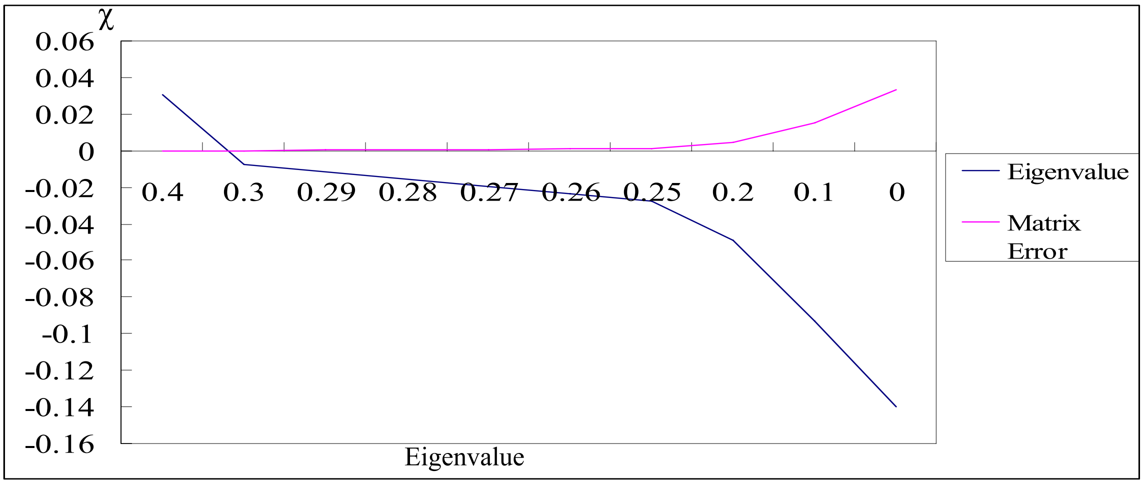

Figure 1: The Error and Eigenvalue of a Matrix using A spectral Decomposition

y-axis is χ ($\chi = \sum_{ij} (C_{ij} - \tilde{C}_{ij})^2$) that is the error of matrix and x-axis is the smallest eigenvalue form the second column of Table 1.

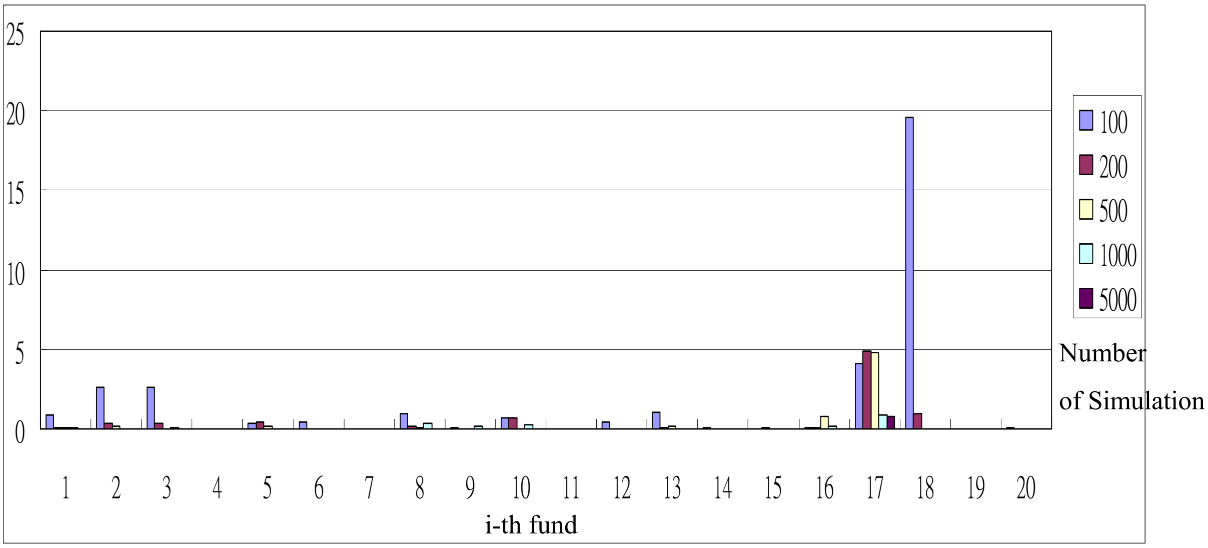

Figure 2: The Numbers of Simulations and the Difference of Two Methods

We define the results of cholesky decomposition and Spectral decomposition after Monte Carlo simulation as *Dc* and *Ds*. Numbers of the simulation are 100, 200, 500, 1000 and 5000. Define error is:

$$error_i = (Dc_i - Ds_i)^2$$

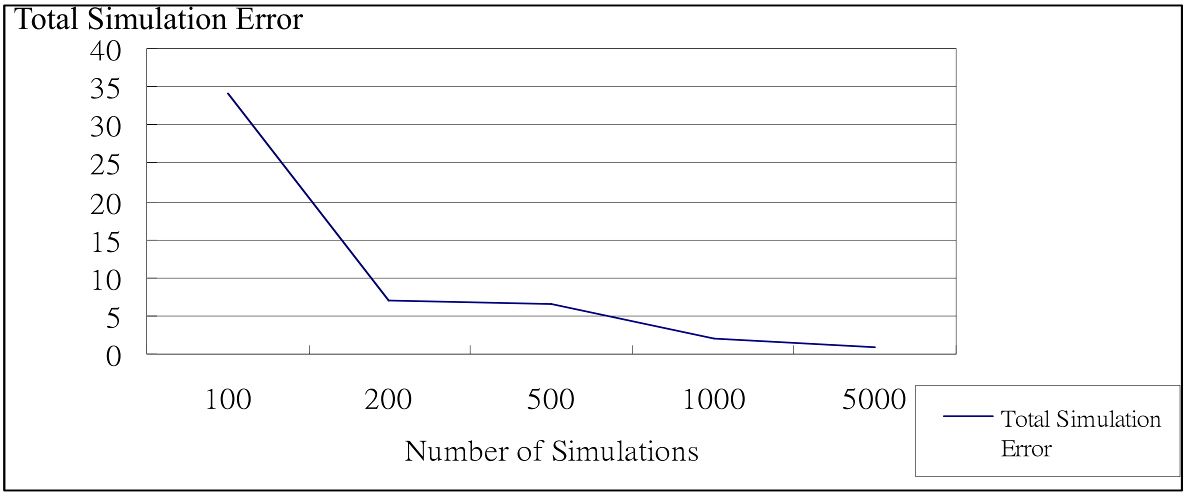

Figure 3: The Numbers of Simulations and the Total Simulation Errors

We define the results of a Cholesky decomposition and a Spectral decomposition using Monte Carlo simulations as *Dc* and *Ds*. The Numbers of the simulation are 100, 200, 500, 1000 and 5000. Define the total error measure of simulations is:

$$\sum_{i=1}^{20} error_i = \sum_{i=1}^{20} (Dc_i - Ds_i)^2$$